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Structural Equivalence of Arithmetical Expressions as a Technical Basis for Philosophical Arithmetic

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Abstract

The word "fracterm" is a portmanteau of "fraction" and "term." Fracterms are instances of arithmetical terms, A-terms, for short. Four notions of correspondence for A-terms are distinguished: syntactic equality, structural equivalence, semantic equivalence (having the same value), and frugal equality. We provide axioms for structural equivalence and discuss the connection with semantic equivalence and frugal equality.

1 Introduction

In [3] we have argued that a "fracterm" – a portmanteau of "fraction" and "term" – may serve as a key concept for elementary arithmetic. The main argument for the use of fracterm instead of fraction is that fraction is an ambiguous notion which, dependent on the context may either denote a syntactic construct, equipped with a numerator, a denominator and an operator symbol for division, or a number which is not plausibly understood as having such components. We intend to describe elementary arithmetic as precisely as possible, and for that purpose having a clear language about fracterms is indispensable.

We will work in the signature of common meadows, as discussed in [12, 13] and further work in [19, 23, 24, 17]. A fracterm is an expression of the form

$$\frac{p}{q}$$

In particular $\frac{1}{0}$ is a fracterm, in spite of its interpretation as $\bot$ which signals some form of error.

We will focus on rational arithmetic. More specifically we assume the presence of three sets of values:

(i) $\mathbb{N}_d$, the set of decimal natural numbers, decimal digit sequences without redundant initial zeroes (e.g. $0, 25, 2035$),

(ii) $\mathbb{Z}_d$, the set of decimal integers, $\mathbb{N}_d$ augmented with $-n$ for nonzero $n \in \mathbb{N}_d$, and

(iii) $\mathbb{Q}_d$, the set of decimal fracrationals, $\mathbb{Z}_d$ augmented with fracterms $\frac{a}{b}$ and $-\frac{a}{b}$ with $a \in \mathbb{N}_d$, $b \in \mathbb{N}_d$, $a \neq 0, b > 1$, and such that $\gcd(a, b) = 1$.

We notice that besides decimal fracrationals one may consider, for instance, unary fracrationals, binary fracrationals, and hexadecimal fracrationals. Further, decimal mixed fracrationals have, in addition to decimal integers, the forms $c\frac{a}{b}$ and $-c\frac{a}{b}$ with a, b, c positive decimal integers, such that $a < b$ and $\gcd(a, b) = 1$. For such numbers $c\frac{a}{b}$ denotes the rational number corresponding to $c + \frac{a}{b}$ and $-c\frac{a}{b}$ denotes the rational number corresponding to $-(c + \frac{a}{b})$. We further notice that decimal rational has a standard meaning as the value of a fracterm the denominator of which is a power of 10. For that reason we speak of decimal fracrationals rather than decimal rationals.

1.1 Four notions of equality, equivalence, and/or correspondence

The proposal to speak of decimal naturals (or decimal natural numbers), rather than of a decimal notation for natural numbers, and correspondingly for decimal integers, stems from [4], where an extensive motivation for this somewhat unusual terminology can be found. These elements are all qualified as so-called numerals. The numerals outside $\mathbb{Z}_d$ are fracterms written in decimal notation, which is to say that both numerator and denominator are decimal integers.

We will use four notions of equality/equivalence/correspondence on arithmetical terms:

(i) We will write $t =_{\mathsf{syn}} r$ to express that t and r are syntactically the same, that is, precisely identical as expressions; we will refer to $=_{\mathsf{syn}}$ as syntactic equality. Syntactic equality works modulo substitution in context and replacement of meta-variables. For instance if $P =_{\mathsf{syn}} 2{+}x$ and context $C[-]$ is given by $C[Q] =_{\mathsf{syn}} \frac{(Q)+1}{3} + y$, then $C[P] =_{\mathsf{syn}} \frac{(2+x)+1}{3} + y$.

(ii) $t \equiv_{\mathsf{str}} r$ expresses that t and r are structurally equivalent, which is a matter of possibly different but essentially equivalent bracketing,

(iii) The somewhat unusual (see remark 5 in the concluding section) notation $t =_{\mathsf{cm}} r$ is used to express that t and r have the same interpretation as a number, or as a function of one or more variables, where values are determined in the common meadow of decimal fracrationals, that is t and r are equivalent (i.e. have the same value),

(iv) Finally $t =^{\mathsf{fr}}_{\mathsf{cm}} r$ denotes so-called frugal equality between open terms as well as closed terms. $t =^{\mathsf{fr}}_{\mathsf{cm}} r$ is a truth-value in a three-valued logic with besides T and F a third truth value U for undefined.

Frugal equality is an option besides two other options for equality-like relations: eager equality $=^{\mathsf{ea}}_{\mathsf{cm}}$ and cautious equality $=^{\mathsf{ca}}_{\mathsf{cm}}$. Eager equality and cautious equality were both defined in [19]. Frugal equality and cautious equality are both non-reflexive, whereas eager equality is not transitive. Eager equality and cautious equality are both relations in a conventional binary logic. The idea of eager equality is to have $t =^{\mathsf{ea}}_{\mathsf{cm}} \bot$ (that is, to consider $t =^{\mathsf{ea}}_{\mathsf{cm}} \bot$ to be true) for all terms t including $\bot$, and the idea of cautious equality is to consider $t =^{\mathsf{ca}}_{\mathsf{cm}} \bot$ to be false for all terms t including $\bot$.

Of the various equality or equality-like relations on rationals we know of, $=^{\mathsf{fr}}_{\mathsf{cm}}$ comes most close to conventional intuitions concerning the meaning of expressions in elementary arithmetic. Nevertheless we do not intend to suggest that users of the equality sign $=$ will have $=^{\mathsf{fr}}_{\mathsf{cm}}$ in mind. Instead we merely view $=^{\mathsf{fr}}_{\mathsf{cm}}$ as the best approximation currently known to us of equality ($=$) as it is used in the practice of elementary arithmetic. In addition, when observing rules concerning mathematical style which prevent an author making use of expressions that have $\bot$ as the interpretation at an outermost level, and when only using quantifiers over non-$\bot$ numbers, guidelines for the prodcuction of arithmetical texts arise that may lead to texts in which no problematic instances of division by zero occur.

1.2 Arithmetical expressions: A-terms

Arithmetical expressions are strings of characters taken from the following set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, -, \cdot, \dot{\bar{:}}, a - z, A - Z, (,), \llcorner\!\lrcorner\}$. Here $-$ denotes both unary negation (opposite value) and two-place subtraction and $\dot{\bar{:}}$ denotes one of the signs for the division operator, while $\llcorner\!\lrcorner$ denotes the option to insert space. We will not pay attention to line breaks which would add a level of complexity which we intend to abstract from in this phase. There are several alternatives for the notation of division, for instance $x \div y = x/y = \frac{x}{y}$.

Identifiers, used as names for new constants or as variable names, are strings consisting of symbols {0,1,2,3,4,5,6,7,8,9,a-z,A-Z} starting with a letter, that is, a character in {a-z,A-Z}.

Expressions are also called terms. We will use A-terms for the class of arithmetical expressions. Fracterms are a subset of A-terms. At the basis of the definition of A-terms are the constants and the identifiers. Constants are the decimal fracrationals (including decimal natural numbers and decimal integers), as outlined above.

Identifiers are, say $a, b, c, x, y, Y, Z, Z23$ etc. as outlined above.

Definition 1.1. *A-terms are inductively defined as follows:*

(i) constants and identifiers are A-terms,

(ii) if t and r are A-terms then $-t, t+r, t-r, t\cdot r, \frac{t}{r}$ and (t) are A-terms. (These operations are called term constructors for A-terms.)

We will not be precise about the occurrence of spacing. If spacing is taken into account as well a clause (iii) must be included:

(iii) if t is an A-term then $\llcorner\!\lrcorner t, t\llcorner\!\lrcorner$ and $\llcorner\!\lrcorner t\llcorner\!\lrcorner$ are A-terms as well.

Proposition 1.1. *An A-term has an even number of brackets, whereby a normal pattern of opening brackets and closing brackets must be respected.*

1.3 Contexts

A context $C[\,]$ is an A-term with a hole in it, where the hole is represented as $[\,]$. For instance in $2+(3-x)\cdot 5$ the minus sign occurs in a subterm $3-x$ in the context $C[\,] =_{\mathsf{syn}} 2 + ([\,]) \cdot 5$. We will say that the minus operator occurs in a $3-x$ subcontext within $C[\,]$. Syntactic equality naturally extends to contexts.

Alternatively one may use contexts with named placeholders. We will use \$1,\$2 etc. as names for holes in a context. Now A-terms can be formed by means of rewriting placeholders. E.g. starting with $C[\$1, \$2, \$3] =_{\mathsf{syn}} \$1 + \$2 - \frac{2\cdot(\$3)+5}{7}$ one may substitute $4+y$ for \$1, -2 for \$2 and $(z+1)$ for \$3 thus obtaining $4 + y + -2 - \frac{2\cdot((z+1))+5}{7}$.

1.4 p/m-terms and t-terms

A-terms will be split in two complementary subsets, the p/m-terms (an abbreviation for plus/min terms) and the t-terms (tight terms). For instance $2-1$ is a p/m-term while $(2-1)$ is a t-term. The p/m-terms are “porous” in the context of multiplication. Consider the context $C[\,] = 5\cdot[\,]$ then $C[2-1] =_{\mathsf{syn}} 5\cdot 2-1$ and rules of calculation involve the higher priority of multiplication over addition/subtraction so that an expression with value 9 results. On the other hand $C[(2-1)] =_{\mathsf{syn}} 5\cdot(2-1)$ yields an expression with value 5.

Definition 1.2. *p/m-terms are A-terms terms with plus (+) or two place minus (−) as one of the possible leading function symbols, depending on parsing. So for t to be a t-term it must either have the form $r+s$ or the form $r-s$ with r and s A-terms (or both forms at the same time).*

Thus a p/m term is an A-term t which either has the form $t =_{\mathsf{syn}} r+s$ or the form $t =_{\mathsf{syn}} r-s$ for some suitable A-terms r and s. It is important to notice that r and s are A-terms and have an even number of brackets, so that the $+$ sign (or $-$ sign) will not be embedded in a bracket pair, with the opening bracket contained in r and the closing braced contained in s, so that indeed the occurrence of $+$ (or $-$) can be understood as an instance of a leading function symbol.

As examples we mention: $(2+3)$ is not a p/m term and neither is $(x-y)$. Further $3+2-5$ is a p/m term in two different ways: $r+s$ with r standing for 3 and s abbreviating $2-5$ or $r-s$ with r abbreviating $3+2$ and s standing for 5.

Further t-terms constitute a notion complementary to p/m-term, that is, t-terms $=$ A-terms $\cap\,\overline{\text{p/m}-\text{terms}}$. The t-terms allow an inductive definition as follows:

- Constants and identifiers are t-terms,
- If t is an A-term then (t) is a t-term,
- If t and r are A-terms then $\frac{t}{r}$ is a t-term,
- If t and r are t-terms then $-t, t\cdot r$, and (t) are t-terms.

As it turns out p/m-terms may be qualified as such on different grounds: for instance consider the A-term $P =_{\mathsf{syn}} 2 + 51 - x + \frac{777}{3} - 1$, then:

- With $R =_{\mathsf{syn}} 2$ and $S \equiv 51 - x + \frac{777}{3} - 1$ we have $P =_{\mathsf{syn}} R + S$, and

- With $R' =_{\mathsf{syn}} 2+51$ and $S' =_{\mathsf{syn}} x + \frac{777}{3} - 1$ we have $P =_{\mathsf{syn}} R' - S'$, and

- With $R'' =_{\mathsf{syn}} 2+51-x$ and $S'' =_{\mathsf{syn}} \frac{777}{3} - 1$: $P =_{\mathsf{syn}} R'' + S''$, and
- With $R''' =_{\mathsf{syn}} 2+51-x+\frac{777}{3}$ and $S''' =_{\mathsf{syn}} 1$: $P =_{\mathsf{syn}} R''' - S'''$.

Proposition 1.2. *Every A-term t is either a p/m-term or a t-term.*

Proof. If A-term t is not a p/m-term, then either it contains no occurrences of plus $(+)$ or two-place minus $(-)$, in which case t is seen to be a t-term by induction on its structure, or all occurrences of $+$ or of two-place $-$ are within a subcontext with an outermost bracket pair, in which case it is also shown with induction on the size of t that it must be a t-term. □

Definition 1.3. *A plusterm is a p/m-term t which cannot be written as $t =_{\mathsf{syn}} r - s$ and a minterm is a p/m-term t which cannot be written as $t =_{\mathsf{syn}} r + s$.*

1.5 Structural equivalence of A-terms

Structural equivalence is an equivalence relation on A-terms. If t and r are structurally equivalent that condition is also referred to as: t and r are equistructural, though we will not use the latter phrase below.

Definition 1.4. *Table 1 defines structural equivalence ($t \equiv_{\mathsf{str}} r$) on A-terms in the sense that structural equivalence is the smallest congruence on A-terms as equipped with the term formation operators which satisfies the equations 10-25.*

Proposition 1.3.
(i) $(t - r) \equiv_{\mathsf{str}} ((t) - r)$,
(ii) $(t - r) \equiv_{\mathsf{str}} (t + (-r))$,
(iii) $(t + (r - s)) \equiv_{\mathsf{str}} (t + r - s)$,
(iv) $((t + r) - s) \equiv_{\mathsf{str}} (t + r - s)$,
(v) $((t - r) + s) \equiv_{\mathsf{str}} (t - r + s)$,
(vi) $(t + (r + s)) \equiv_{\mathsf{str}} (t + r + s)$,
(vii) $((t + r) + s) \equiv_{\mathsf{str}} (t + r + s)$,
(viii) $(t + (r) + s) \equiv_{\mathsf{str}} (t + r + s)$.

Proof. These proofs are easily done with the combination of rules 18,19, and 20 of Table 1.

Moreover, we notice that whenever $t \equiv_{\mathsf{str}} t'$ and $r \equiv_{\mathsf{str}} r'$ also: $-(t) \equiv_{\mathsf{str}} -(t'), t + r \equiv_{\mathsf{str}} t' + r', t - r \equiv_{\mathsf{str}} t' - r', t \cdot r \equiv_{\mathsf{str}} t' \cdot r'$, and $\frac{t}{r} \equiv_{\mathsf{str}} \frac{t'}{r'}$. As a consequence the equivalence relation $\equiv_{\mathsf{str}}$ is a congruence w.r.t. the term constructors. □

Proposition 1.4. *An A-term t satisfies $t \equiv_{\mathsf{str}} (t)$ if and only if it is a t-term.*

Proof. If $t \equiv_{\mathsf{str}} (t)$ then t cannot be a p/m-term because all rewrite rules of Table 1 applied in either way, when applied to (t) will preserve the form (s) with s a p/m-term. The other direction of the assertion is proven by

$$t \equiv_{\mathsf{str}} t \tag{1}$$
$$t \equiv_{\mathsf{str}} r \to r \equiv_{\mathsf{str}} t \tag{2}$$
$$t \equiv_{\mathsf{str}} r \wedge r \equiv_{\mathsf{str}} s \to t \equiv_{\mathsf{str}} s \tag{3}$$
$$t \equiv_{\mathsf{str}} t' \to (t) \equiv_{\mathsf{str}} (t') \tag{4}$$
$$t \equiv_{\mathsf{str}} t' \to -t \equiv_{\mathsf{str}} -t' \tag{5}$$
$$t \equiv_{\mathsf{str}} t' \wedge r \equiv_{\mathsf{str}} r' \to t + r \equiv_{\mathsf{str}} t' + r' \tag{6}$$
$$t \equiv_{\mathsf{str}} t' \wedge r \equiv_{\mathsf{str}} r' \to t - r \equiv_{\mathsf{str}} t' - r' \tag{7}$$
$$t \equiv_{\mathsf{str}} t' \wedge r \equiv_{\mathsf{str}} r' \to t \cdot r \equiv_{\mathsf{str}} t' \cdot r' \tag{8}$$
$$t \equiv_{\mathsf{str}} t' \wedge r \equiv_{\mathsf{str}} r' \to \frac{t}{r} \equiv_{\mathsf{str}} \frac{t'}{r'} \tag{9}$$

$$((t)) \equiv_{\mathsf{str}} (t) \tag{10}$$
$$(\nu) \equiv_{\mathsf{str}} \nu \text{ for a numeral } \nu \tag{11}$$
$$(\alpha) \equiv_{\mathsf{str}} \alpha \text{ for an identifier } \alpha \tag{12}$$
$$-(\alpha) \equiv_{\mathsf{str}} -\alpha \text{ for an identifier } \alpha \tag{13}$$
$$\frac{(t)}{r} \equiv_{\mathsf{str}} \frac{t}{r} \tag{14}$$
$$\frac{t}{(r)} \equiv_{\mathsf{str}} \frac{t}{r} \tag{15}$$
$$(\frac{t}{r}) \equiv_{\mathsf{str}} \frac{t}{r} \tag{16}$$
$$(-(t)) \equiv_{\mathsf{str}} -(t) \tag{17}$$
$$t + -r \equiv_{\mathsf{str}} t - r \tag{18}$$

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$$(t + r) \equiv_{\mathsf{str}} ((t) + r) \tag{19}$$
$$(t + r) \equiv_{\mathsf{str}} (t + (r)) \tag{20}$$
$$((t) \cdot (r)) \equiv_{\mathsf{str}} (t) \cdot (r) \tag{21}$$
$$(t) + (r) + (s) \equiv_{\mathsf{str}} (t) + r + (s) \tag{22}$$
$$(t) + u + (r) + (s) \equiv_{\mathsf{str}} (t) + u + r + (s) \tag{23}$$
$$(t) + (r) + v + (s) \equiv_{\mathsf{str}} (t) + r + v + (s) \tag{24}$$
$$(t) + u + (r) + v + (s) \equiv_{\mathsf{str}} (t) + u + r + v + (s) \tag{25}$$

Table 1: Rules for structural equivalence of A-terms

induction on the size of t-terms. We need a case distinction on decompositions of t depending on the leading function symbol. We know that neither $+$ nor two place $-$ can serve as a leading function symbol for t, as otherwise t is a p/m-term. We will only consider the case of multiplication, the other cases being even simpler. Now suppose $t \equiv_{\mathsf{str}} r \cdot s$ because t is not a p/m-term both r and s are not p/m-terms, i.e. are t-terms. Thus $r \equiv_{\mathsf{str}} (r)$ and $s \equiv_{\mathsf{str}} (s)$ so that $t \equiv_{\mathsf{str}} r \cdot s \equiv_{\mathsf{str}} (r) \cdot (s) \equiv_{\mathsf{str}} ((r) \cdot (s)) \equiv_{\mathsf{str}} (t)$. □

Definition 1.5. *An A-term t is fully bracketed if the following conditions are satisfied for t:*

- Every occurrence of $-$ is either in a subcontext of the form $((t)-(r))$, or alternatively in a context of the form $(-(t))$,

- Every occurrence of $+$ is in a subcontext of the form $((t)+(r))$,

- Every occurrence of $\cdot$ is in a subcontext of the form $((t)\cdot(r))$.

We make use of the vertical notation for fracterms by assuming that $\frac{t}{r}$ is a t-term. Were we to use a conventional infix notation for division, for example $t \div r$, then $t \div r$ is not always a t-term and the definition ought to include an additional requirement that each occurrence $\div$ must be positioned in a subcontext of the form $((t) \div (r))$.

Proposition 1.5. *For every t-term t there is a t-term r in fully bracketed form such that $t \equiv_{\mathsf{str}} r$.*

Proof. Let t be some given t-term. By Proposition 1.4 $t \equiv_{\mathsf{str}} (t)$, and with induction on the structure of terms t we prove for all s that there is a t-term r in fully bracketed form such that $(s) \equiv_{\mathsf{str}} r$. For s an identifier or a decimal numeral for rational numbers (that is, a simplified simple fracterm with decimal integers as its numerator and denominator) it suffices to take $r \equiv_{\mathsf{str}} s$ and then to use rules 11 resp. 12 of Table 1. For the induction step we consider the case of two-place minus. Suppose $s \equiv_{\mathsf{str}} s' - s''$. Using the induction hypothesis there are fully bracketed t-terms r' and r'' such that $(s') \equiv_{\mathsf{str}} r'$ and $(-s'') \equiv_{\mathsf{str}} r''$. We find that $(s) \equiv_{\mathsf{str}} (s' - s'') \equiv_{\mathsf{str}} (s' + -s'') \equiv_{\mathsf{str}} ((s') + -s'') \equiv_{\mathsf{str}} ((s') + (-s'')) \equiv_{\mathsf{str}} (r' + r'') \equiv_{\mathsf{str}} ((r') + r'') \equiv_{\mathsf{str}} ((r') + (r''))$. The latter A-term is fully bracketed. □

We insist that $t \equiv_{\mathsf{str}} r$ implies semantic equality ($\equiv$) in the common meadow of rationals for t and r.

The rules for $\equiv_{\mathsf{str}}$ allow to minimize the number of brackets. Minimization is done under the implicit assumption that addition and multiplication are associative. For instance working within t-terms Proposition 1.3 items (vi) and (vii) combined yield the associativity of addition.

(A complication is that for instance $1 + (2+3) \not\equiv_{\mathsf{str}} 1 + 2 + 3$. Indeed, consider right multiplication with 4 then $1 + (2+3) \cdot 4$ evaluates to 21 while $1 + 2 + 3 \cdot 4$ evaluates to 15, so that $1 + (2+3) \cdot 4 \neq 1 + 2 + 3 \cdot 4$ and also $1 + (2+3) \cdot 4 \not\equiv_{\mathsf{str}} 1 + 2 + 3 \cdot 4$. Similarly $1 + 2 \not\equiv_{\mathsf{str}} 2 + 1$.

In a context $C[\ ]$ the brackets in a subcontext $\$1 \cdot (\$2 + \$3)$ or in a subcontext $(\$1 + \$2) \cdot \$3$ are necessary. Thus with $C[\ ] =_{\mathsf{syn}} 3 + [\ ] \cdot y$, $\$1 =_{\mathsf{syn}} a$, $\$2 =_{\mathsf{syn}} b - 3$, and $\$3 =_{\mathsf{syn}} Z$ we find that the brackets in $C[3 + (a + b - 3) \cdot Z \cdot y]$ cannot be removed.

More generally if r is a p/m-term then the bracket pair in $t \cdot (r)$ cannot be removed, neither can it in $(r) \cdot t$. The rules 22-25 allow the removal of redundant brackets in p/m-terms. The idea of these rules is to guarantee that transforming a p/m-term t with these rules within a context $C[t \cdot (r)]$ and $C[(r) \cdot t]$ will not lead to wrong order of evaluation of addition and multiplication.

1.6 Intuition and motivation of structural equivalence

The intuition and motivation for the introduction and definition of A-terms and structural equivalence of A-terms is as follows:

(i) Given that addition and multiplication are associative, structural equivalence captures the different forms of bracketing for A-terms that are essentially the same.

(ii) However, in order to understand the definition of A-terms and of structural equivalence of A-terms an understanding of the concept, as well as formalities, of associativity, is not required as a prerequisite.

(iii) When setting up a syntax for elementary arithmetic, a plausible approach is first to have addition as a two-place operator and then to argue associativity thereby justifying say $1+2+3$ as a somehow simplified rendering of $(1 + 2) + 3$ or of $1 + (2 + 3)$. What speaks against such an approach, however, is that when introducing arithmetic one prefers to allow expressions like $1+2+3$ right from the start, without being bothered by bracketing and associativity. The definition of A-terms implements the said preference.

(iv) In [4] we experimented with an introduction to arithmetical syntax which adopts for each $n \geq 2$ an n-ary addition operator. Axioms are then given for relating the different operators for different arities.

(v) Unfortunately, and perhaps unexpectedly, the approach of [4] turns out to be rather complicated, for instance by involving an infinite signature, and by the use of overloaded syntax for the different addition operators. In particular the mixed occurrence of two-place addition and two-place subtraction creates a complicated signature where for instance $f(x, y, z) =_{\mathsf{def}} x + y - z$ and $g(x, y, z) =_{\mathsf{def}} x - y + z$ are distinguished as three-place functions. Indeed a naive approach in line with abstract datatype theory involves a combinatorial explosion concerning the number of functions. These complications are such that looking for an alternative approach is called for, if not necessary.

(vi) We feel that working with A-terms and structural equivalence overcomes the complications of [4] in a plausible manner.

2 Fracterm calculus: semantic models

Fracterm calculus is a reference to elementary arithmetic in which all fracterms are recognized as relevant syntactic entities for which a meaning must be determined. We will discuss four different fracterm calculi. Elements of an arithmetical datatype which do not correspond to an element of the conventional number system are called peripheral numbers.

(i) For common meadows (based on adopting $\frac{1}{0} = \bot$), which we consider to be the preferred option. Upthorn, $\bot$, is the unique peripheral number of a common meadow.

(ii) For arithmetic with Suppes-Ono division (using $\frac{1}{0} = 0$), featuring no peripheral element.

(iii) For transrational arithmetic (working with $\frac{1}{0} = +\infty$), which involves three peripheral elements ($+\infty, -\infty$ and $\bot$), and

(iv) The wheel of rational numbers where one adopts $\frac{1}{0} = \infty$, an unsigned form of infinite peripheral number. A wheel also contains an absorptive element $\bot$.

Structural equality can be used in the context of each of the three other cases. In case of Suppes-Ono division $\bot$ is not needed, and may be deleted from the syntax. In case of transrational arithmetic an additional constant ∞ is used which represents positive infinity. The additional fact in connection with syntax is: $(\infty) =_{\mathsf{str}} \infty$. Said additional fact comes as a consequence of Table 1 upon understanding ∞ as a numeral, which is moderately plausible.

The logic of fracterms, or rather of A-terms, in each of the three cases, involves working with equations. Reasoning with equations involves substitution: suppose that $t =_{\mathsf{cm}} r$ is a valid equation which involves a variable x. Now once $t =_{\mathsf{cm}} r$ has been established and given some term u also $[(u)/x]t =_{\mathsf{cm}} [(u)/x]r$ is established. Here $[P/x]$ is the transformation of terms which substitutes P for all occurrences of x. For instance, given the equation $x + y =_{\mathsf{cm}} y + x$, let $u =_{\mathsf{syn}} 17 - z$ we may infer $(17 - z) + y =_{\mathsf{cm}} y + (17 - z)$. In this particular example the bracketing is unproblematic, and in fact $[u/x]t =_{\mathsf{cm}} [u/x]r$ is valid (reading $17 - z + y =_{\mathsf{cm}} y + 17 - z$). However, consider the equation $x \cdot y =_{\mathsf{cm}} y \cdot x$. While $(17 - z) \cdot y =_{\mathsf{cm}} y \cdot (17 - z)$ is valid, that is not the case for $17 - z \cdot y =_{\mathsf{cm}} y \cdot 17 - z$. In this latter case, substitution of $17 - z$ for x requires the preparatory packaging of the p/m-term (in fact minterm) $17 - z$ in the t-term $(17 - z)$.

2.1 Four domains

The four models we have in mind share most of their domains, as each of these models consists mainly of decimal fracrationals as discussed above. The common meadow of (decimal) rationals, $\mathbb{Q}^d_\bot$ contains in addition the element $\bot$ as a peripheral number. The involutive meadow of decimal numbers $\mathbb{Q}^d_0$ contains precisely the decimal fracrationals and has no peripheral elements. The decimal transrationals feature three peripheral elements: $+\infty$, $-\infty$, and $\bot$. The wheel of decimal fracrationals involves two peripherals: ∞ and $\bot$. We notice that in transarithmetic $\bot$ is denoted Φ (named nullity), and has a philosophical status somewhat different from $\bot$. As in [18], however, we will use $\bot$ for the unique absorptive element in order to obtain a more uniform treatment of different fracterm calculi.

2.2 Common meadows: using common division

We will write $=_{\mathsf{cm}}$ for (semantic) equality in fracterm calculus for common meadows. Fracterm calculus for common meadows works well in the case

$$(t) =_{\mathsf{cm}} t \qquad \text{for all A-terms t} \tag{26}$$
$$t \equiv_{\mathsf{str}} r \to t =_{\mathsf{cm}} r \qquad \text{for all A-terms t and r} \tag{27}$$

$$(x+y)+z =_{\mathsf{cm}} x+(y+z) \tag{28}$$
$$x+y =_{\mathsf{cm}} y+x \tag{29}$$
$$x+0 =_{\mathsf{cm}} x \tag{30}$$
$$x+(-x) =_{\mathsf{cm}} 0 \cdot x \tag{31}$$
$$x \cdot (y \cdot z) =_{\mathsf{cm}} (x \cdot y) \cdot z \tag{32}$$
$$x \cdot y =_{\mathsf{cm}} y \cdot x \tag{33}$$
$$1 \cdot x =_{\mathsf{cm}} x \tag{34}$$
$$x \cdot (y+z) =_{\mathsf{cm}} (x \cdot y) + (x \cdot z) \tag{35}$$
$$-(-x) =_{\mathsf{cm}} x \tag{36}$$
$$0 \cdot (x \cdot x) =_{\mathsf{cm}} 0 \cdot x \tag{37}$$
$$x + \bot =_{\mathsf{cm}} \bot \tag{38}$$

Table 2: $E_{\mathsf{wcr},\bot}$ axioms for weak commutative rings with $\bot$

of so-called common division, that is, upon assuming

$$\frac{x}{0} =_{\mathsf{cm}} \bot$$

Axioms for common meadows are listed in Tables 2 and 3. These axioms are a minor, though logically equivalent, variation on the axioms given in [12]. In that paper inversive notation was used, for these axioms in divisive notation we mention [18].

2.3 Involutive meadows: using Suppes-Ono division

Suppes-Ono division adopts $\frac{x}{0} = 0$ for all x. To the best of our knowledge, adopting $\frac{1}{0} = 0$ has been the first suggestion for division by zero which has been worked out in some noticeable detail, we have traced back such work to Patrick Suppes 1957, with a corresponding logical analysis in [27]; see also [2] for an account of Suppes' approach. The equations of Table 4 axiomatize the class of involutive meadows (inverse is an involution). These axioms are taken from [9]. The subscript of $=_{\mathsf{so}}$ abbreviates "Suppes-Ono", a convention which was not yet adopted in [9], however.

$$\texttt{import: } E_{wcr,\bot}$$

$$x =_{\mathsf{cm}} \frac{x}{1} \tag{39}$$

$$\frac{x}{y} \cdot \frac{u}{v} =_{\mathsf{cm}} \frac{x \cdot u}{y \cdot v} \tag{40}$$

$$\frac{x}{y} + \frac{u}{v} =_{\mathsf{cm}} \frac{(x \cdot v) + (y \cdot u)}{y \cdot v} \tag{41}$$

$$\frac{x}{y + 0 \cdot z} =_{\mathsf{cm}} \frac{x + 0 \cdot z}{y} \tag{42}$$

$$\bot =_{\mathsf{cm}} \frac{1}{0} \tag{43}$$

Table 3: $E_{\mathsf{ftc-cm}}$: equations for fracterm calculus for common meadows

$$(t) =_{\mathsf{so}} t \qquad \text{for all A-terms t} \tag{44}$$

$$t \equiv_{\mathsf{str}} r \to t =_{\mathsf{so}} r \qquad \text{for all A-terms t and r} \tag{45}$$

$$(x + y) + z =_{\mathsf{so}} x + (y + z) \tag{46}$$

$$x + y =_{\mathsf{so}} y + x \tag{47}$$

$$x + 0 =_{\mathsf{so}} x \tag{48}$$

$$x + (-x) =_{\mathsf{so}} 0 \tag{49}$$

$$x \cdot (y \cdot z) =_{\mathsf{so}} (x \cdot y) \cdot z \tag{50}$$

$$x \cdot y =_{\mathsf{so}} y \cdot x \tag{51}$$

$$x \cdot 1 =_{\mathsf{so}} x \tag{52}$$

$$x \cdot (y + z) =_{\mathsf{so}} (x \cdot y) + (x \cdot z) \tag{53}$$

$$\frac{1}{(\frac{1}{x})} =_{\mathsf{so}} x \tag{54}$$

$$\frac{x \cdot x}{x} =_{\mathsf{so}} x \tag{55}$$

$$\frac{x}{y} =_{\mathsf{so}} x \cdot \frac{1}{y} \tag{56}$$

Table 4: Axioms for involutive meadows with divisive notation

$$(t) =_{\mathsf{tr}} t \qquad \text{for all A-terms t} \tag{57}$$

$$t \equiv_{\mathsf{str}} r \to t =_{\mathsf{tr}} r \qquad \text{for all A-terms t and r} \tag{58}$$

Table 5: Additional axioms for transrational arithmetic

$$(t) =_{\mathsf{wh}} t \qquad \text{for all A-terms t} \tag{59}$$

$$t \equiv_{\mathsf{str}} r \to t =_{\mathsf{wh}} r \qquad \text{for all A-terms t and r} \tag{60}$$

Table 6: Additional axioms for wheel arithmetic

2.4 Transrational arithmetic: signed explosive division

Transreals and transrationals are due to [1]. For a recent account, see [26]. Equational axioms for transrational arithmetic can be found in [15] where 36 axioms are used for an initial algebraic specification of the datatype of transrational numbers. We will not repeat these axioms. In line with the approach taken before, and denoting equality for transrationals with $=_{\mathsf{tr}}$ two rules must be additionally included, as displayed in Table 5.

We propose to use *explosive division* for forms of division where $\frac{1}{0}$ produces an infinite value, or rather a peripheral value representing one or more intuitions of infinity. Transrational arithmetic features a signed explosive division as it produces a signed infinity.

2.5 Wheel arithmetic: unsigned explosive division

Wheels were introduced in [22]. Equational axioms for a wheel of rationals can be found in [16], where equations are used for an initial algebra specification of the datatype of a wheel of rationals. We will not repeat these axioms here. In line with the approach taken before, and denoting equality for transrationals with $=_{\mathsf{wh}}$ two rules must be additionally included, as displayed in Table 6.

2.6 Values and evaluation

In each of the four mentioned arithmetical datatypes the elements of the domain constitute a subset of A-terms, where the set of constants for peripheral entities varies from 0 to 3. The idea is that all values are closed A-terms but not all closed A-terms are values. In this section we assume that each closed A-terms "has" a value, meaning that it is equal (for a common meadow in the sense of $=_{\mathsf{cm}}$ etc.) to a number in the arithmetic at hand. The notation for the value of a closed A-term t in arithmetical datatype M is: $(M \models t)$. Alternatively we may use $\mathsf{val}_M(t)$. If a term involves free variables, evaluation requires the presence of a valuation σ, and the corresponding notations are $(M, \sigma \models t)$ and $\mathsf{val}_{M,\sigma}(t)$. In practice at large, however, these notations are never used. Instead one assumes that a context is known in terms of a choice that has been made for an arithmetical datatype so that this choice may be left implicit. Moreover it is unfamiliar to use explicit notations for term evaluation because, as much as possible, a term stands ambiguously both for itself and for its meaning, that is, for the resulting value upon evaluation.

Now let, for instance, $\mathbb{Q}_\perp^d$ be chosen as a context for (elementary) arithmetical work, for the three other options, as listed above, a similar account can be given. The simplest questions one may pose are about closed equations: suppose closed A-terms P and Q are given, the question "do P and Q have the same value" is about the validity of the equation $t =_{\mathsf{cm}} r$ in $\mathbb{Q}_\perp^d$, and this question can be answered by determining the (unique) values, say p and q, respectively, of both P and Q, and then deciding whether or not these are identical (i.e. $p =_{\mathsf{syn}} q$).

1. The value of $\frac{3 \cdot 4}{5 \cdot 2}$ is $\frac{6}{5}$.
2. The fracterm $\frac{3 \cdot 4}{5 \cdot 2}$ can be simplified while the fracterm $\frac{6}{5}$ cannot be simplified. Note that $\frac{6}{5}$ is both a fracterm and a number.
3. If a fracterm P can be simplified to Q, it will be the case that, as a result of simplification, P and Q are different, because both $P \neq_{\mathsf{syn}} Q$ and $P \not\equiv_{\mathsf{str}} Q$.
4. The value of $\frac{2}{1}$ is 2.
5. The A-term $\frac{3}{1}$ is a fracterm which can be simplified, but the unique result of simplification is not a fracterm, it is a decimal natural number (and a constant A-term for that reason).
6. Addition of fracterms is a general label for several notions each involving the wish to preserve a certain form. Clearly for all fracterms P and Q, $P + Q =_{\mathsf{cm}} \frac{P+Q}{1}$, so that the claim that fracterms can be added with the result again in the form of a fracterm is vacuous by triviality.
7. Addition for fracterms is a relation rather than a mathematical function. Like most practical tasks addition more often than not can be done in different ways thereby obtaining a result which satisfies a certain condition (just like travelling from A to B in a given timeframe, or like preparing a meal). For instance:

   - One may wish to add flat fracterms in such a manner as to obtain a flat fracterm, (flat fracterms a fracterms for which both numerator and denominator do not contain any division operators),

   - Or one may wish to add simple fracterms in such a manner as to obtain a simple fracterm,

   - Or one may wish to add simplified simple fracterms in such a manner as to obtain a simplified simple fracterm (in general undoable, however: consider $\frac{1}{2} + \frac{1}{2} =_{\mathsf{cm}} 1$; now 1 is not the value of any simplified simple fracterm, although it is the result of simplification of a simple fracterm viz. $\frac{1}{1}$).

8. We consider "ratio" to be a notion different from fracterm, and also different from rational number or decimal fracrational. For positive values a and b, $a : b$ denotes the ratio of a to b. We write $a : b \equiv c : d$ if (and only if) $a \cdot c = b \cdot d$. We hold that the four place relation "$a : b \equiv c : d$" constitutes the only relevant syntax for ratios and that in spite of its appearance $a : b$ must not be understood as an A-term. Instead the word ratio merely serves as a label for a pair which one intends to use in a certain setting, that is in claims and conclusions of the form "$a : b \equiv c : d$".

2.7 Compositional evaluation of closed fracterms

We will write $\mathsf{val}(t)$ for $\mathsf{val}_{\mathbb{Q}^d_\perp}(t)$. It is not entirely straightforward to explain evaluation (that is determination of $\mathsf{val}(t)$) for closed fracterms in compositional terms.

To begin with: $\mathsf{val}(p) =_{\mathsf{syn}} p$ for decimal fracrationals p and $\mathsf{val}(\perp) =_{\mathsf{syn}} \perp$ and $\mathsf{val}((t)) =_{\mathsf{syn}} \mathsf{val}(t)$, moreover $\mathsf{val}(t + r) =_{\mathsf{cm}} \mathsf{val}(t) + \mathsf{val}(r)$. $\mathsf{val}(t \cdot r) =_{\mathsf{cm}} \mathsf{val}(t) \cdot \mathsf{val}(r)$. If both t and r are t-terms then $\mathsf{val}(t \cdot r) =_{\mathsf{cm}} \mathsf{val}(t) \cdot \mathsf{val}(r)$, and more generally for all A-terms t and r, $\mathsf{val}((t) \cdot (r)) =_{\mathsf{cm}} \mathsf{val}((t)) \cdot \mathsf{val}((r)) =_{\mathsf{cm}} \mathsf{val}(t) \cdot \mathsf{val}(r)$.

However, determination of $\mathsf{val}(t \cdot r)$ and of $\mathsf{val}(r \cdot t)$ is less obvious if r is a p/m-term. Indeed, with $t =_{\mathsf{syn}} 2$ and $r =_{\mathsf{syn}} 3 + 1$ one finds $\mathsf{val}(t \cdot r) =_{\mathsf{syn}} \mathsf{val}(2 \cdot 3 + 1) =_{\mathsf{cm}} 7$ while $\mathsf{val}(t) \cdot \mathsf{val}(r) =_{\mathsf{syn}} \mathsf{val}(2) \cdot \mathsf{val}(3 + 1) =_{\mathsf{cm}} 8$. A similar issue arises with negative values: $\mathsf{val}(-3 + 1) \neq_{\mathsf{cm}} -\mathsf{val}(3 + 1)$. Compositionality is achieved, however, for fully bracketed terms.

3 Frugal equality

Arithmetic appears in textual expositions in which it is used, explained, or discussed, including educational texts. By merely designing or selecting an arithmetical datatype one can neither discover nor explain the requirements constraining such texts. Texts can be generated via grammars and additional constraints may then be used to filter out acceptable texts.

An option for developing such constraints is to disallow any expressions which have value $\perp$ (at an outermost level) and to use quantifiers which range over non-$\perp$ elements only. By doing so a reasonable correspondence with mathematical practice may be achieved. Of course such conventions only matter in the presence of $\perp$, but when using Suppes-Ono division, $\perp$ does not enter the picture to begin with, so that additional issues concerning division by zero do not arise in that case.

Frugal equality, written $t =^{\mathsf{fr}}_{\mathsf{cm}} r$ in case of common meadows, $t =^{\mathsf{fr}}_{\mathsf{tr}} r$ in case of transrationals, and $t =^{\mathsf{fr}}_{\mathsf{wh}} r$ in case of wheels, is a relation which is slightly less informative than $=_{\mathsf{cm}}$, $=_{\mathsf{tr}}$, and $=_{\mathsf{wh}}$ respectively. Frugal equality makes use of a three valued logic involving U as a third truth value. Frugal equality (or equalities) comes (come) with a short-circuit logic, rather than a classical logic and with quantifiers $\forall^{\mathsf{f}}$ and $\exists^{\mathsf{f}}$ which quantify over all elements with the exception of $\perp$.

We will make some comments regarding $t =^{\mathsf{fr}}_{\mathsf{cm}} r$ below, see also Table 7 where some connections between these congruences are listed. For closed

$$\forall^{\mathsf{f}} x.(x =^{\mathsf{fr}}_{\mathsf{cm}} x) \tag{61}$$

$$\forall x.(0 \cdot x =_{\mathsf{cm}} 0 \to t - r =_{\mathsf{cm}} 0) \to \forall^{\mathsf{f}} x.(t =^{\mathsf{fr}}_{\mathsf{cm}} r) \tag{62}$$

$$0 \cdot (x + y) =_{\mathsf{cm}} 0 \wedge x =_{\mathsf{cm}} y \to x =^{\mathsf{fr}}_{\mathsf{cm}} y \tag{63}$$

Table 7: Some rules linking frugal equality with common meadow equality

A-terms t and r:
$t =^{\mathsf{fr}}_{\mathsf{cm}} r$ takes value T if $t \neq_{\mathsf{cm}} \bot$ and $r \neq_{\mathsf{cm}} \bot$ and $t =_{\mathsf{cm}} r$,
$t =^{\mathsf{fr}}_{\mathsf{cm}} r$ takes value F if if $t \neq_{\mathsf{cm}} \bot$ and $r \neq_{\mathsf{cm}} \bot$ and $t \neq_{\mathsf{cm}} r$, and
$t =^{\mathsf{fr}}_{\mathsf{cm}} r$ takes value U if $t =_{\mathsf{cm}} \bot$ or $r =_{\mathsf{cm}} \bot$.

A typical assertion in the logic thus obtained is ϕ given by

$$\forall^{\mathsf{f}} x.(x \neq^{\mathsf{fr}}_{\mathsf{cm}} 0 \mathbin{\circ\!\!\!\rightarrow} \frac{x}{x} =^{\mathsf{fr}}_{\mathsf{cm}} 1)$$

We notice that ϕ is valid, in the sense that $x \neq^{\mathsf{fr}}_{\mathsf{cm}} 0 \mathbin{\circ\!\!\!\rightarrow} \frac{x}{x} =^{\mathsf{fr}}_{\mathsf{cm}} 1$ is true for all substitutions of a non-$\bot$ value for x. What ϕ achieves, however, is to return T for the substitution $x = 0$ while not requiring that $\frac{0}{0} =^{\mathsf{fr}}_{\mathsf{cm}} 1$ is either true or false. Using frugal equality the truth value of $\frac{0}{0} =^{\mathsf{fr}}_{\mathsf{cm}} 1$ becomes U. The latter is due to the logic at hand which adopts $\mathsf{F} \mathbin{\circ\!\!\!\rightarrow} \mathsf{U} = (\neg \mathsf{F}) \mathbin{{}^{\circ}\!\vee} \mathsf{U} = \mathsf{T} \mathbin{{}^{\circ}\!\vee} \mathsf{U} = \mathsf{T}$. The 3-valued logic thus obtained is quite nonclassical as conjunction and negation are not commutative: $\mathsf{F} \mathbin{{}_{\circ}\!\wedge} \mathsf{U} = \mathsf{F}$ while $\mathsf{U} \mathbin{{}_{\circ}\!\wedge} \mathsf{F} = \mathsf{U}$ and $\mathsf{T} \mathbin{{}^{\circ}\!\vee} \mathsf{U} = \mathsf{T}$ while $\mathsf{U} \mathbin{{}^{\circ}\!\vee} \mathsf{T} = \mathsf{U}$.

3.1 Frugal equality for other models of division

Frugal equality works similarly when starting out from transrational equality. For closed A-terms t and r:
$t =^{\mathsf{fr}}_{\mathsf{tr}} r$ takes value T if $t \neq_{\mathsf{tr}} \bot$ and $r \neq_{\mathsf{tr}} \bot$ and $t =_{\mathsf{tr}} r$,
$t =^{\mathsf{fr}}_{\mathsf{tr}} r$ takes value F if if $t \neq_{\mathsf{tr}} \bot$ and $r \neq_{\mathsf{tr}} \bot$ and $t \neq_{\mathsf{tr}} r$, and
$t =^{\mathsf{fr}}_{\mathsf{tr}} r$ takes value U if $t =_{\mathsf{tr}} \bot$ or $r =_{\mathsf{tr}} \bot$.

Similar clauses can be given for $=^{\mathsf{fr}}_{\mathsf{wh}}$ in the case of wheel arithmetic:
$t =^{\mathsf{fr}}_{\mathsf{wh}} r$ takes value T if $t \neq_{\mathsf{wh}} \bot$ and $r \neq_{\mathsf{wh}} \bot$ and $t =_{\mathsf{wh}} r$,
$t =^{\mathsf{fr}}_{\mathsf{wh}} r$ takes value F if if $t \neq_{\mathsf{wh}} \bot$ and $r \neq_{\mathsf{wh}} \bot$ and $t \neq_{\mathsf{wh}} r$, and
$t =^{\mathsf{fr}}_{\mathsf{wh}} r$ takes value U if $t =_{\mathsf{wh}} \bot$ or $r =_{\mathsf{wh}} \bot$.

Frugal equality $=^{\mathsf{fr}}_{\mathsf{so}}$ for Suppes-Ono division is just $=_{\mathsf{so}}$ and offers no novel perspective.

3.2 Frugal equality and short-circuit logic

Frugal equality is attractive in the case of common meadows. The logic thus obtained is conditional short-circuit logic as discussed in detail and axiomatized in [14] using the primitives of the proposition algebra of [11] with the notation of [7]. Said short-circuit logic is equivalent to the conditional logic of [25]. In case of transarithmetic we also expect some simplification of the logic by suppressing the use of $\perp$ but only to a lesser extent as infinite values still violate conventional laws such as distribution of multiplication over addition.

A connection between the conditional equational logic of common meadows and the conditional short-circuit logic of meadows is as follows: a conditional equation $t =_{\mathsf{fr}} r \circ\!\!\rightarrow u =_{\mathsf{fr}} v$ involving variables say x and y has truth value T (out of three options) in all common meadows (though working with frugal equality) if and only if the conditional equation $t = r \wedge 0 \cdot (t + r) = 0 \rightarrow u - v = 0$ is true in all common meadows. This idea trivially generalizes to arbitrarily many conditions and variables.

3.3 Eager equality and cautious equality

In [19] we have discussed eager equality and cautious equality. Like frugal equality these relations are variants of equality dealing with $\perp$ in a particular manner. Eager equality can be introduced for each model of division: $=^{\mathsf{ea}}_{\mathsf{cm}}$, $=^{\mathsf{ea}}_{\mathsf{so}}$, $=^{\mathsf{ea}}_{\mathsf{tr}}$, and $=^{\mathsf{ea}}_{\mathsf{wh}}$, where $=^{\mathsf{ea}}_{\mathsf{so}}$ is just the same as $=_{\mathsf{so}}$.

Likewise cautious equality can be introduced for each model of division: $=^{\mathsf{ca}}_{\mathsf{cm}}$, $=^{\mathsf{ca}}_{\mathsf{so}}$, $=^{\mathsf{ca}}_{\mathsf{tr}}$, and $=^{\mathsf{ca}}_{\mathsf{wh}}$, where $=^{\mathsf{ca}}_{\mathsf{so}}$ is once more the same as $=_{\mathsf{so}}$.

Eager equality and cautious equality are not quite equality relations and rules of logical deduction must be adapted in both cases. The underlying assumption of eager equalities is that $\perp$ is considered equal with any element, in other words: $x =^{\mathsf{ea}} \perp$, and more specifically: $x =^{\mathsf{ea}}_{\mathsf{cm}} \perp$, $x =^{\mathsf{ea}}_{\mathsf{tr}} \perp$, and $x =^{\mathsf{ea}}_{\mathsf{wh}} \perp$. In a complementary fashion the assumption underlying cautious equality is that every element, including $\perp$ differes from $\perp$: $x \neq^{\mathsf{ca}}_{\mathsf{cm}} \perp$, $x \neq^{\mathsf{ca}}_{\mathsf{tr}} \perp$, and $x \neq^{\mathsf{ca}}_{\mathsf{wh}} \perp$.

4 Naming classes of fracterms

In [3] we have discussed the conventional naming for classes of fracterms (usually referred to as classes of fractions), e.g. common fracterms, proper fracterms, unit fracterms, flat fracterms, composed fracterms, open fracterms, and closed fracterms, however, some additional detail is needed.

4.1 Conventional terminology for fracterms

From [3] we recall naming conventions for the following properties of fracterms:

1. Simple fracterm: a decimal fracterm $\frac{p}{q}$ with p a decimal integer and $q > 0$ a decimal natural; a simple fracterm may also be referred to as common (potentially confusing in the context of common meadows) or vulgar (in older textbooks but now consdered to be outdated),

2. The decimal fracterm $\frac{p}{q}$, with $p > 0$ and $q > 1$ decimal integers, can be simplified if for some natural $r > 1$, r divides p and r divides q,

3. With the notation of the above item: if $p = r \cdot p'$ and $q = r \cdot q'$ and $q > 1$ then $\frac{p'}{q'}$ is a result of simplification of $\frac{p}{q}$; moreover, if

$p = r \cdot p'$ and $q = r$ and $q > 1$ then the decimal natural p' is a result of simplification of $\frac{p}{q}$,

4. A simple fracterm is simplified if it cannot be simplified,
5. Proper fracterm: a decimal fracterm $\frac{p}{q}$ with p and q decimal naturals and such that $0 < p < q$,
6. Unit fracterm: a unit fracterm is a fracterm of the form $\frac{1}{p}$ with p a decimal natural above 1,
7. Flat fracterm: a flat fracterm is a fracterm of the form $\frac{p}{q}$ where p and q are terms without any occurrence of the division operator,
8. Composed fracterm: a composed fracterm is a fracterm which is not flat.

4.2 Additional terminology for fracterms

To the conventional terminology we add the following possible novel terminology:

1. A fracterm of the form $\frac{t}{0}$ is a fracterm with null-denominator, or alternatively a null-denominated fracterm,
2. a flat fracterm of the form $\frac{t}{r}$ with r closed where r evaluates to the integer 0 (i.e. $r =_{\mathsf{cm}} 0$) is a fracterm with a zero-valued denominator (clearly a fracterm with a null-denominator is also a fracterm with a zero-valued denominator, but not the other way around),
3. A flat fracterm with a zero-valued denominator is alternatively called essentially null-denominated,
4. A flat fracterm of the form $\frac{t}{r}$ with r closed where r evaluates to a non-zero integer (that is, $r \neq_{\mathsf{cm}} 0$) is a fracterm with a nonzero-valued denominator,
5. A fracterm of the form $\frac{t}{1}$ is said to be trivial,
6. A fracterm of the form $\frac{t}{r}$ with r closed where r evaluates to 1 is said to be essentially trivial,
7. A fracterm of the form $\frac{t}{r}$ with r closed where r evaluates to an integer different from 0 and from 1 is said to be essentially non-trivial.

5 Methodological considerations

We list some methodological remarks in arbitrary order.

1. We view this work as a contribution to philosophical arithmetic as discussed in [6]. In the terminology of [6] the perspective on division above is formal, rather than prescriptive or prospective. The contribution we claim lies in achieving a high level of precision for writing about elementary arithmetic. Specific issues in this area are:
   - What is the semantics of the equality sign (=) in elementary arithmetic?
   - What is the relation between logic and elementary arithmetic?

- Is there a best or even standard notion of legality? (We refer to [20] for the requirement of legality in elementary arithmetic.)
- "Fraction" is quite ambiguous, in any case "fraction" is more ambiguous than "fracterm" and more ambiguous than "rational number". Not distinguishing a fracterm from the rational number it denotes may be compared to not distinguishing a Cauchy sequence from the real number it determines, and may be a source of confusion. Now the question is: is there a need, in the context of elementary arithmetic, to make use of highly ambiguous terminology?
- If no need to use highly ambiguous terminology can be established: is there a significant advantage in using ambiguous terminology, and if not: is there and advantage to be obtained from abandoning ambiguous terminology as much as possible (for instance by using fracterm instead of fraction whenever fracterm is meant, and using rational number or value whenever the value of a fracterm is meant.)

2. One of the tasks for philosophical arithmetic is to analyse how equality (=) is used in elementary arithmetic. None of the relations $=_{\mathsf{cm}}$, $=_{\mathsf{so}}$, $=_{\mathsf{tr}}$, $=_{\mathsf{wh}}$, $=^{\mathsf{fr}}_{\mathsf{cm}}$, $=^{\mathsf{fr}}_{\mathsf{tr}}$, $=^{\mathsf{fr}}_{\mathsf{wh}}$, $=^{\mathsf{ea}}_{\mathsf{cm}}$, $=^{\mathsf{ea}}_{\mathsf{tr}}$, $=^{\mathsf{ea}}_{\mathsf{wh}}$, $=^{\mathsf{ca}}_{\mathsf{cm}}$, $=^{\mathsf{ca}}_{\mathsf{tr}}$, and $=^{\mathsf{ca}}_{\mathsf{wh}}$ meets such demanding requirements. We consider these relations first of all to be tools that may be helpful for investigating the practice of elementary arithmetic.

3. Plusterms were named sumterms in [4]. The approach to sumterms (that is, plusterms) of [4] is rather complex because it involves the introduction of an infinite number of so-called long additions (additions of arity above 2). We feel that the above approach to plusterms is simpler.

4. Structural equivalence ($\equiv_{\mathsf{str}}$) as defined with Table 1 is of relevance for elementary arithmetic rather than for arithmetic in general. Associativity of addition and multiplication is presumed in the design of $\equiv_{\mathsf{str}}$, and when describing a version of arithmetic that features non-associative addition or non-associative multiplication, other, less abstract, forms of structural equivalence may be considered.

   Non-associative operations may arise with the enlargement of common meadows with peripheral numbers for infinite entities, in particular peripherals for describing overflow in finitary versions of arithmetic, or conceptual (infinite) models thereof.

   By deleting rules 19 to 25 from Table 1 one obtains a description of strict structural equivalence $\equiv_{\mathsf{str/s}}$ which may serve as a preparation for setting up an arithmetic with non-associative addition and/or multiplication.

5. We have renamed equality of fracterm calculus for common meadows as introduced in [12] by $=_{\mathsf{cm}}$. The idea of this particular renaming is not so much that after all, we consider $=^{\mathsf{fr}}_{\mathsf{cm}}$ to be a better choice for the core relational primitive of fracterm calculus than =. Instead we view fracterm calculus as a module in the sense of module algebra as defined in [8], which in certain applications may be transformed by

way of renaming (one of the primitives of module algebra) in order to obtain a better fit with the context of the application.

The application we have in mind in the current paper is to provide meta-theory for elementary school arithmetic, as well as to suggest options for educational use. In other applications we may prefer to use arithmetic with common division with the ordinary equality sign.

6. The naive fracterm calculus of [20] describes the informal logic of elementary arithmetic where at best casual notice is taken of the contrast between syntax and semantics. The synthetic fracterm calculus of [21] suggests an informal approach to elementary arithmetic where, at least in principle, the methods of first order logic are made available. The definitions of the current paper may be used as a technical elaboration of synthetic fracterm calculus.
7. Working with frugal equality and a corresponding three-valued logic can be understood as a formalization of working with division as a partial function which takes no value when dividing by zero. When modelling division as a partial function the use of a three-valued logic with sequential logic connectives is quite plausible.
8. One may doubt the rationale of the formation of a term $3+x\cdot 5+2$ as $t \cdot r$ with $t =_{\mathsf{syn}} 3+x$ and $r =_{\mathsf{syn}} 5+2$. However, we hold that whosoever intends to denote the multiplication of $3+x$ and $5+2$ ought to be aware that for that purpose $t' \cdot r'$ with $t' =_{\mathsf{syn}} (3+x)$ and $r' =_{\mathsf{syn}} (5+2)$ is appropriate.

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