Transmathematica DOI Prefix 10.36285 ISSN 2632-9212

Transreal Foundation
For Floating-Point Arithmetic

James A.D.W. Anderson
james.a.d.w.anderson@btinternet.com
88 Lower Henley Road, Caversham, Reading, England, RG4 5LE

Submitted: 13 June 2023
Revised: 23 July 2023

Abstract

Software Engineering relies, to a large extent, on formal software standards and logical means for specifying and verifying computer programs. Among these the IEEE 754 standard for floating-point arithmetic is widely used. We criticise the standard from the standpoint of transreal arithmetic. Transreal arithmetic was derived from projective geometry using a double cover to provide signed infinities on the horizon and nullity at the point of projection. These infinities and nullity have some similarities with IEEE 754 floating-point infinities and NaNs but there are important differences. We explore the differences by analysing the standard at three levels: commentary within the standard, abstract datatypes, and bit patterns. We find that all of the differences are coincident with faults in the standard. Obviously a correct standard would better support the specification, development and testing of numerical software. We discuss how the standard can be corrected, in its own terms, or by adopting transreal arithmetic as its theoretical foundation. We also discuss emulation of transreal arithmetic in IEEE 754 processing systems and address accusations that transreal arithmetic plagiarised the standard.

1 Introduction

Numerical software enables large parts of our technological society so it is important that it can be correctly specified and verified. The IEEE standard for floating-point arithmetic [1] [2] [3] is very widely used in numerical software and in the design of the computer processors that support almost all general purpose computing so it is of interest to the majority of programmers and end users. The standard formalises certain numerical computing practices in the computing industry. While many parts of the standard have a technical motivation, there is no uniform theory of arithmetic underlying the standard, which has lead to the faults discussed here. We propose that transreal arithmetic can provide a coherent foundation for the development of correct numerical standards and software, including the development of transreal emulators implemented in IEEE 754 floating-point processing systems, and the eventual development of trans-floating-point hardware.

We begin by summarising the historical development and mathematical properties of transreal arithmetic. This provides a logical and consistent foundation from which to criticise the standard and gives the software engineer access to the emerging technical literature on transreal solutions to various theoretical and practical problems.

We proceed by criticising the 1985, 2008 and 2019 versions of the standard in order. As the standards are backwardly compatible, in the parts that are of interest, this is both an economical order of work and one which highlights the development of the standard over time, which then suggests how future improvements might be incorporated.

We criticise the standard at three levels. Firstly we criticise the commentary within the standard and suggest both how the existing wording could be clarified and how the standard could be expressed more clearly in terms of transreal arithmetic. Secondly we criticise two abstract datatypes within the standard, being the zeros and the Not-a-Number (NaN) objects. These are all deeply problematical floating-point values that, as we shall prove, render the standard both non-logical and inconsistent. Hence they prevent the specification and verification of floating-point programs at the level of abstract data, but allow them at the level of bit patterns. Thirdly we criticise the standard at the level of bit patterns. We propose changes to the processing of floating-point bits so that all abstract floating-point values are both logical and consistent. We illustrate such changes by rehearsing radical proposals to implement trans-floating-point [4] and trans-two's complement [5] arithmetic. We also criticise the recommended mathematical functions and the methodology for defining them set out in the 2008 and 2019 versions of the standard. We briefly describe the practice of emulating correct transreal behaviour in the standard's floating-point arithmetic. This emulation provides correctness at the cost of slower execution and passing up calculation to a tolerance twice as fine as the standard's arithmetic in the same number of bits.

Finally we rebut unsupported accusations [67] that transreal arithmetic plagiarised the IEEE standard and consider what evidence would be needed to establish that claim.

2 Transreal Arithmetic

This paper is written in the third person or passive voice when it discusses matters of public record but is written in the first person active voice when it discusses private information that explains key steps in the public development of transmathematics.

Transreal numbers and their arithmetic were developed from 1997 to 2007. Thereafter the transreal numbers served as the foundation for the development of a wider transmathematics which aims to totalise the whole of contemporary mathematics, notwithstanding that some parts of contemporary mathematics are already total.

The *point at nullity* was introduced into projective geometry at a workshop on human and computer vision in 1997, [6]. At that time, in that community, it was the practice to puncture projective space by removing the projection point and discarding it. I followed the practice of puncturing projective space but retained the projection point and named it the point at nullity. This gives the point at nullity a distinct topology from the finite and infinite points that make up projective space. I was aware that the infinite points on the horizon have homogeneous co-ordinates with the syntactic form $\frac{x}{0}$, with $x \neq 0$, and the point at nullity, at the projection point, has syntactic form $\frac{0}{0}$, but I followed the contemporary practice of using geometrical constructions and matrix operations to solve problems in computer vision without explicitly dividing by zero.

In 1999, at a conference on numerical methods, the transrational numbers [7] were introduced in their explicit form: *negative infinity,* $-\infty = \frac{-1}{0}$; *nullity,* $\Phi = \frac{0}{0}$; *positive infinity,* $+\infty = \frac{1}{0}$. The transarithmetical operations of addition, subtraction, multiplication, reciprocal, and division were defined by operating on the syntactic form of a fraction $\frac{n}{d}$, now called a *fracterm* [8] [9], where the numerator, n, is an arbitrary integer and the denominator, d, is a positive or zero integer. I carefully selected the syntactic operations from the various contemporary algorithms for fractional arithmetic so that division by zero is permitted and all of the usual results of rational arithmetic are preserved.

That paper contains a trivial proof that transrational addition, subtraction, and multiplication are consistent with rational arithmetic and that the reciprocal is the only possible source of inconsistency.

The ordering of the transrational numbers was calculated using the relationship $a > b \iff a-b > 0$, whence nullity is unordered with respect to any other transrational number, the rational numbers have their usual ordering, positive infinity is greater than any other ordered transrational number, and negative infinity is less than any other ordered transrational number. Notice that transrational ordering was not defined axiomatically, nor was it defined by consensus and a formal vote in any IEEE committee, it was derived by transarithmetical calculation.

That paper also used an ancient Greek formula, from Euclid's Elements, to define rational versions of some trigonometric functions that map from a transrational half tangent to a transrational value. Hence all of the transrational numbers appear as both arguments and as values of a function. This establishes all of the transrational syntactic fractions, or fracterms, as numbers for two reasons. Firstly they are numbers for the classical reason that they arise as points in a geometrical, specifically a trigonometric, construction. Secondly they are numbers for the modern reason that they arise as solutions to an equation. This numberhood then applies to the geometrical constructions and syntactic homogeneous coordinates of projective geometry, discussed above. However, there are now two choices of how to define infinity. We can have an unsigned infinity, $\infty = -\infty = +\infty$, as occurs in single cover projective geometries, or we can have a signed infinity, $-\infty < +\infty$, as occurs in both trigonometry and double cover projective geometries. As a signed infinity gives us both Euclidean trigonometry and projective geometry, I took it is as the canonical form of infinity.

At this stage $-\infty, \Phi, +\infty$ were established as numbers for geometrical and algebraic reasons, prior to their appearance as limits, though the usual practice was followed of writing ∞ as a synonym for $+\infty$. Thus transrational arithmetic is logically prior to both transreal arithmetic and transreal analysis.

Notice that nullity is not the metalogical symbol bottom, $\perp$. This is easy to prove. Let $f(\frac{n}{d}) = n$ be a function which returns the numerator of its argument, where the argument, n/d, is a transreal number in canonical form. Then $f(\Phi) = 0$ but $f(\perp) = \perp$. However, we cannot establish what $f(\text{NaN})$ evaluates to, unless and until the relevant IEEE committee establishes a consensus and takes a formal vote to settle the matter.

Transrational arithmetic was developed further in a series of conferences dedicated to the geometry of vision, with an emphasis on computer vision. We now cite the more salient of these papers.

In 2002, transrational arithmetic [10] adopted the rational numbers, together with just two non-finite numbers, nullity and positive infinity, $+\infty > 0$. This allowed trigonometric functions to obtain infinity as a number and negative infinity as an asymptotic limit but not a number. In 2005, transreal numbers, being the real numbers together with nullity and positive infinity, were introduced [11]. In 2006, the transreal numbers were extended by readopting negative infinity and adopting what is now called *tetrachotomy*: every transreal number is exactly one of (1) less than zero, (2) equal to zero, (3) greater than zero, (4) equal to nullity [12]. At this stage I realised that having signed infinities, $-\infty < 0 < +\infty$, can provide an unsigned infinity as the absolute value, $\infty = |\pm\infty|$, so no expressive power is lost by adopting signed infinities – and some simplification is obtained by treating negative infinity as a number, not an asymptotic limit. At this stage transreal arithmetic was in its final form.

In 2007 a machine proof of the consistency of transreal arithmetic was given [13]. During the preparation of that paper, I realised, in discussion with my co-authors, that signed infinities are needed in calculus to provide positive infinity and negative infinity as numbers to which limits can be taken; thus in the fragment, $\int_{-\infty}^{\infty}$, the syntactic objects $-\infty, \infty$ are not symbols denoting a limiting process running over real numbers, they are transreal numbers, $-\infty = \frac{-1}{0}$, $\infty = \frac{1}{0}$, to which transreal limits are taken. Henceforth I maintained the view that transreal arithmetic is the least change from real arithmetic that allows division by zero and supports the whole of real analysis. So far I am not aware of any other total arithmetic with these properties [14] [15].

The machine proof of the consistency of transreal arithmetic [13] gained world-wide publicity. Thereafter some people claimed, without giving evidence, that transreal arithmetic plagiarised IEEE floating-point arithmetic. We consider this claim [67] in more detail below but the historical evidence is that transreal arithmetic was based on projective geometry, trigonometry, calculus, and syntactic algorithms, as just documented.

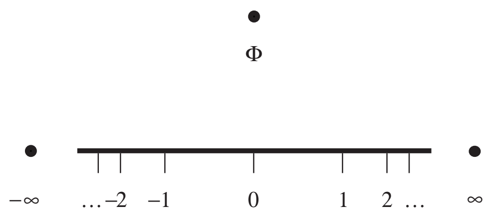

Figure 1: Transreal number line.

The transreal exponential and logarithm were defined in [16]. These are consistent with transreal multiplication and division. They also show how certain trigonometric functions can be given as power series and how this agrees with their construction as geometrical figures. In the usual extended real arithmetics the four terms $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \infty - \infty, 0^0$ are all undefined but in transreal arithmetic, $\frac{0}{0}$ is defined to be nullity, Φ, whence it is a theorem that $\frac{\pm\infty}{\pm\infty} = \infty - \infty = 0^0 = \Phi$. Thus these four terms are all well defined in transreal arithmetic. That paper also discussed how to remove infinities from a formula, say by using the term 1^x which is unity for all real x and nullity for all non-finite transreal x. In particular this extends various real trigonometric identities so that they hold for all transreal values; for example $\cos^2 x + \sin^2 x = 1^x$.

In the technical literature it has been known since 1957 that division by zero is possible [15] but [16] explicitly dissolves various popular "proofs" that division by zero is impossible.

In 2008, the topology of transreal numbers was calculated transarithmetically from epsilon neighbourhoods [5]. Thus transreal arithmetic provides metrics and defines its own set topology, hence it defines its own geometry.

The whole of the transreal number line is shown in the finite space of Figure 1. The solid line is the whole of the real number line. The centre of the finitely long line, in the figure, is marked zero. The midpoint between zero and the right-hand end of the line is marked one. The midpoint between this point and the right-hand end of the line is marked two. The dieresis, '. . . ', indicates that this marking is carried out over the entire sequence of natural numbers. The missing real numbers are then filled in between each pair of natural numbers including zero. The right-hand end is topologically open so the marking iterates without end. Positive infinity is shown as a discrete point on a rightward extension of the real number line. The gap between the right-hand open end of the real number line and the closed positive infinity is empty in the transreal numbers but may contain the transfinite infinities of other number systems. This is to say that transreal positive infinity is greater than all other transfinite infinities and is, in fact, the greatest number. This requires careful handling in some set theories. The left-hand side of the figure is marked similarly, which completes the extended real number line with a greatest and least infinity. Nullity lies off this line and is conventionally shown above zero. This topology defines the ordering of all of the numbers on the extended real number line and defines that nullity is unordered because it is a singleton point that lies off that line. Recall that the topology of the transreal number line, including its ordering, is calculated transarithmetically.

That paper recommends using this topology in saturated two's complement arithmetic such that zero and all opposite pairs of integers, $(i, -i)$, occupy their usual bit positions but the most negative, unpaired integer, known as the *weird number,* is replaced with nullity, while the adjacent pair of integers, being the remaining greatest and least integer, are replaced, respectively, with positive and negative infinity, thereby saturating the arithmetic. If wraparound is to be permitted then it should be done in a way that preserves transreal topology. As usual the operation of forming the complement of a number models negation but, in two's complement arithemtic, the weird number has the erroneous behaviour that its negation remains identically negative and does not appear as its positive opposite. By contrast the complement of nullity is correctly given as nullity because, in transreal arithmetic, $-\Phi = -\frac{0}{0} = \frac{-0}{0} = \frac{0}{0} = \Phi$. This is a very important point. Two's complement provides an erroneous model of a finite subset of real arithmetic but a correct model of a finite subset of transreal arithmetic. Put another way, taking real arithmetic as a foundation for two's complement arithmetic introduces faults but taking transreal arithmetic as a foundation does not – it is fault free.

Note that any computer arithmetic, based on transreal arithmetic, has a topology which embeds in the transreal number line, including its gaps. See Figure 1. This provides a common core of rounding modes and machine errors which may inform numerical analysis and code compilation.

In 2011, a preliminary form of the transcomplex numbers was introduced [17].

In 2013, the first paper on transreal arithmetic appeared [18] that was written by authors unconnected to the author of the present paper. These authors then established a fruitful working relationship with me.

In 2014, the transcomplex numbers were given in their current form [19] as polar co-ordinates (r, θ) of a transreal radius, r, and transreal angle θ. Polar form is necessary because Cartesian co-ordinates are generally degenerate at infinity. This degeneracy blocks the Cayley-Dickson construction of hypercomplex numbers but, perhaps, a polar form of the construction can be developed? That paper gives a human proof that the transcomplex numbers are consistent if the complex numbers are. This implies that the transreal numbers are consistent if the real numbers are. But this leads to a conclusion which may surprise the naive reader. Real and complex arithmetic do not allow division by zero, yet both transreal and transcomplex arithmetic allow division by zero and are consistent relative to real and complex arithmetic – *so no argument employing division of real or complex numbers by zero can show that division by zero is impossible.*

In [20], topological sets and measures were used to extend real limits and real continuity to transreal limits and transreal continuity. In particular the transreal tangent and arctangent were defined for all transreal values, such that real values of the tangent retain their usual periodicity but with a different periodicity for infinite and nullity values. The periodicities of the non-finite values of the tangent agree with both their geometrical and power series constructions. In that paper it was proposed to replace real arithmetic with transreal arithmetic to obtain the advantages of totality.

The results in [20] were extended in [21] by defining a transreal derivative and integral. Hence transreal calculus operates exactly at singularities, not just in the asymptotic approach to singularities, thereby providing results that real calculus cannot. Hence it is possible to test transreal functions for continuity at a singularity by checking whether or not the value at the singularity is equal to the limit taken asymptotically to the singularity. This is of interest in physics because functions that are continuous across singularities might support physical motion through a physical singularity, which can also happen by other means [22].

The 1985 and 2008 versions of the IEEE standard were criticised in [4] where it was proposed to replace floating-point arithmetic with trans-floating-point arithmetic. Trans-floating-point arithmetic takes the IEEE bit patterns [1] [2] [3] but replaces minus zero with nullity, moves positive and negative infinity to the extremal bit patterns with all bits set in the exponent and significand, and replaces all NaN objects with real numbers. This adds almost one binade. Incrementing the exponent bias by one takes this binade small so that the the smallest magnitude non-zero number has half the magnitude of the corresponding number in the standard. Thus trans-floating-point arithmetic has a tolerance that is twice as fine as the standard's floating-point arithmetic in the same number of bits. Hence an end user obtains an advantage even if the non-finite transreal numbers do not occur during execution of an application. The trans-floating-point comparisons less than, equal to, greater than, and their negations are mutually exclusive. Ordering of all trans-floating-point numbers is well defined.

In [23] transreal arithmetic was proposed as a foundation for paraconsistent logics using $-\infty$ as absolutely False, 0 as equally False and True, $+\infty$ as absolutely True, and Φ as neither False nor True. The logical operators of many paraconsistent logics can then be given as transarithmetical operators, with arithmetical bounds on the reliability of conclusions.

In 2015, the first Ph.D. in transmathematics was awarded [24]. This established foundational results in transreal topology and analysis.

Also in this year, improved definitions of the transreal derivative and integral were given [25] such that transreal calculus allows division by zero and contains real calculus as a proper subset. This paper also corrected typographical errors in an earlier description of transreal ordering [4]. In [26] the real elementary functions were extended to transreal elementary functions that allow division by zero. Taken together these results mean that transreal analysis allows division by zero and is a proper superset of real analysis. Hence preferring real analysis is questionable, if not perverse, because real analysis cannot match the expressive power of transreal analysis and fails where transreal analysis does not. Put another way, real analysis buys failure at the price of being less expressive. This is a lose-lose situation.

In [27], a transreal logical space was described which allows logical transformations to be applied simultaneously to all logical propositions. In [28], it was proved that within this space there is an infinitely dense set of hypercyclic universal worlds, each of which approximates all worlds in the space by repeated mechanical application of the backward shift operator. In [29] infinitely scaleable pipeline machines were described that employ transreal arithmetic so that no hardware or software exception handling is needed for logical exceptions, though physical faults within the machine must still be handled. In [22] it was shown that transreal arithmetic extends Newton's Laws of Motion and Gravitation so that these laws operate exactly at singularities. This makes division by zero physically meaningful and allows transphysics to make predictions where physics cannot.

In 2016, the topology of the transcomplex numbers was described, whence the complex elementary functions were extended to transcomplex elementary functions [30]. A geometrical construction of all transreal angles was given in terms of windings on the unit cone. The angles $\theta = -\infty, \phi, \infty$ co-occur at the apex of the cone. This agrees with the calculation of trigonometric functions both by geometrical construction and by their transpower series.

In a reversal of the historical order in which results were obtained, [31] gave a constructive proof that the transreal numbers are consistent if the real numbers are. This paper also extended the algebraic structure of a field to a transfield. It was shown that, just as the rationals are the smallest ordered field and reals are the unique ordered complete field, so, under suitable conditions, transrationals are the smallest ordered transfield and transreals are the smallest ordered complete transfield. Thus transfields mirror fields but allow division by zero. That paper noted that historical controversies in mathematics ended when constructive proofs were given and expressed the hope that the controversy surrounding the transreal numbers would now end.

Also in this year, one paper set out the proper and improper (Trans) Riemann Integral in a single definition [32] and another applied transreal arithmetic to a construction in classical logic [33].

In 2017, transcomplex analysis was developed further [34]. In this year the first international conference on transmathematics was held, sponsored by UNESCO, but a publisher withdrew an offer to publish a transmathematics journal with the proceedings and invited longer papers in the first issue. I then established Transmathematica as an online archival journal and published the proceedings in 2019, [35].

In 2018, the first Master's thesis in transmathematics was awarded [36].

In 2019, the second conference on transmathematics and total systems took place [37]. The following were among the papers published. An author [14], unrelated to the author of the present paper, reviewed various approaches to division by zero and introduced the algebraic structure of a premeadow which generalises an associative transfield. A stronger transreal integral was introduced in [38] that is defined on the entire set of transreal numbers and integrates all functions which are properly or improperly integrable on real numbers. In [39], a technique called *slipstreaming* was described and illustrated an implementation of the Fast Fourier Transform (FFT) on a fine-grained architecture that exploits the exception-free properties of transreal arithmetic to implement a statically assigned systolic dataflow. The paper [40] continued investigations into totalising set theories. In [41], the concept of a thought experiment was formalised in a transreal logical space which was used to analyse the Einstein, Podolsky and Rosen paradox (the EPR paradox).

Elsewhere the concepts of trans-Boolean logic and transvectors were defined in a proof of the existence of universal possible worlds [42]. The transcomplex integral was defined in [43].

In 2020 it was shown that involutive meadows can be simplified to a greater extent than transrationals [44]. The datatype of fractions, including rationals, wheels, meadows, and transrationals, was considered in [9]. In [45], the transrationals were further developed as a datatype and there was some discussion of the multiple motivations for the development of transreal arithmetic, including as a model of floating-point arithmetic. A third party review of transreal arithmetic was cited in [46].

In 2021, a critical paper [47] showed that when Dedekind defined the real numbers, in terms of cuts of a line, he excluded three cuts, which prevents division by zero. When these three cuts are included we obtain transreal arithmetic such that the previously excluded cuts operate as transreal $-\infty, \Phi, +\infty$. Now that it is recognised that real and transreal arithmetic have the same basis, in cuts of a line, it is questionable, if not perverse, to exclude three cuts and enforce failure on division by zero. This has very wide ramifications for our society, which is now seen to base much of its mathematics, science and technology on a perverse number system, when it could adopt the total system of transreal arithmetic and avoid failure.

Also in this year, the third international conference on total systems took place [48]. Among the papers was a constructive proof that the transreals are consistent if the hyperreals are [49].

Elsewhere the transcomplex integral was developed further [50].

In 2022, the fourth international conference on total systems took place [51]. Among the presentations, James Anderson introduced transets, using a particular four-valued logic to establish membership of the inside, outside, border, and background of a transet. With this arrangement, sets are transets with an empty border. Anderson showed that all three-valued logics fail because they are partial, whence four-valued logics are the smallest candidate for a total logic that deals with truth, falsehood, and contradiction. Anderson mapped the trans-Dedekind construction of the transreals onto transets and showed that nullity is the transet with all real numbers in its border and everything else in its background. In other words, nullity is the real antinomy. This natural expression of the trans-Dedekind cut suggests that transets are a more natural basis for transreal arithmetic than sets and might serve as a better foundation for mathematics.

Also in this year a review of approaches to division by zero [52] discussed fracterm calculus and the transreals. A paper on totalising partial algebras [53] discussed the transreals, among other approaches to division by zero. The transquaternions and their arithmetic were defined in polar form [54] with all co-ordinates transreal.

In 2023, explicit constructions were given for both discrete and continuous hypercyclic universal worlds in the transreal logical space of all possible worlds [55] and some philosophical implications of this were discussed.

Since the spread of transmathematics to Brazil, a number of publications on transmathematics appeared in Portuguese. Some of these were derived from publications in English: [18] [56] [57] [58] [24] [59] [60] [61] [62] [63] [36] [64] [65] [66].

This concludes our brief summary of the development of transreal arithmetic as a logical and consistent mathematical theory. This summary gives the software engineer access to both theoretical and practical transarithmetical solutions which are guaranteed to be total, that is, they are guaranteed to give correct results in all conceivable logical circumstances, though they can fail due to erroneous implementation and physical faults. No such guarantee can be given to software based on real arithmetic nor, as we shall see, on IEEE floating-point arithmetic.

We are now ready to criticise the industrial practice of floating-point arithmetic as described in the IEEE standard.

3 1985 Standard

In this section we criticise the 1985 version of the IEEE standard [1]. All page numbers refer to that version. This is the only version of the standard that was in scope during the development of transreal arithmetic.

Page 6 and 8, together, say there are three kinds of infinity. Let us say ∞ is an *abstract* infinity which exists only in the commentary of the standard. This is represented by two distinct *concrete* infinities: $+\infty$ is *positive* infinity, with sign bit zero, and $-\infty$ is *negative* infinity, with sign bit unity. Transreal arithmetic also has a positive and negative infinity but their properties are not identical to any of the infinities in the standard.

Page 6 says a signalling NaN must be provided but transreal arithmetic is total so it has no NaN objects and hence has no need to signal their vacuous presence.

Page 7 says a quiet NaN must be provided but transreal arithmetic is total so it has no NaN objects and is, therefore, trivially quiet.

Page 8 says there are three kinds of zero. Let us say that an *abstract* zero, 0, exists only within the commentary of the standard. This is represented by two *concrete* zeros: 0^+ has sign bit zero, whereas 0^- has sign bit unity. Transreal arithmetic has only one zero but its properties are not identical to any of the zeros in the standard.

This page also says that a floating-point format represents a NaN object when its exponent takes on a certain reserved value and the fractional part is non-zero and the sign bit is ignored. Hence there are $2^{p+1} - 2$ NaN bit patterns, where p is the precision of the floating-point format. Note that 2^{p+1} is a large number and corresponds to one binade. Let us say NaN_i is an *abstract* NaN, which exists only in the commentary of the standard, and that it has two *concrete* representations: NaN_i^+ has sign bit zero and NaN_i^- has sign bit unity. Here i is an integer index in the range $0 \leq i \leq 2^p - 2$. Ignoring sign makes $2^p - 1$ states redundant, which corresponds to almost half a binade. Transreal arithmetic has no NaN objects but it does have one nullity, Φ. Replacing the NaN objects with real numbers would add $2^{p+1} - 2$ numbers, almost one binade, as discussed in the previous section.

Page 11 says that the floating-point arithmetical operations of addition, subtraction, multiplication, division, and remainder apply to any two operands of the same format. Hence these operations are apparently total. However, the remainder operation $x \,\mathrm{REM}\, y$ is not defined for $y = 0$ so the remainder operation is partial, not total. The remainder operation has not, so far, been defined in transreal arithmetic; though transreal division by zero is exact so it would be reasonable to define that the remainder is zero when $y = 0$.

Page 14 requires four mutually exclusive comparisons: *less than*, $<$; *equal*, $=$; *greater than*, $>$; *unordered*, ?. Some combinations of these comparisons and their negations are signalling and some are quiet. Every abstract NaN must compare unordered to everything, including itself; whence every concrete NaN must compare unordered to everything, including itself. The concrete zeros must compare equal to themselves and each other so: $0^+ = 0^+$, $0^- = 0^-$, and $0^+ = 0^-$. Hence the abstract zero compares equal to itself $0 = 0$.

Pages 15 and 16 give a table of 14 "functionally distinct useful" combinations of the four mutually exclusive comparisons, whereas we expect $2^4 = 16$ combinations. The two missing comparisons are the empty comparison, ϵ, which has no occurrences of less than, equal, greater than, unorderd; and the full comparison $?<=>$, which has all occurrences of less than, equal, greater than, unordered.

The table gives 12 logical negations, $\mathrm{NOT}(x)$, of some of the 14 comparisons. The missing negations of these 14 comparisons are $\mathrm{NOT}(=)$ and $\mathrm{NOT}(?<>)$. Together with $\mathrm{NOT}(\epsilon)$ and $\mathrm{NOT}(?<=>)$ there are 26 comparisons or negations and 6 missing comparisons or negations.

Page 14 says that $\mathrm{NOT}(x = y)$ and $x? <> y$ are identical, which is to say this comparison and negation are not mutually exclusive. Other comparisons and negations are not identical only by virtue of forming a pair of signalling and quiet versions.

Transreal arithmetic has three comparisons: *less than, equal to, greater than,* but does not have an unordered comparison. All combinations of the transreal comparisons exist and are mutually exclusive, hence all combinations of their negations exist and are mutually exclusive. Furthermore the totality of all non-empty comparisons and their negations is mutually exclusive. In addition the empty comparison and its negation can be given side effects so that they are also mutually exclusive [4] [25]. Hence all transreal comparisons are functionally distinct and useful. Because the transreal comparisons and their negations are total, there are no signalling comparisons or negations. Hence adopting the transreal comparisons and their negations would remove signalling and resolve the missing comparisons and negations in the standard.

The ordering of NaN objects has a devastating consequence. The Law of Identity asserts that for all x it is the case that $x = x$. But $\mathrm{NaN}_0^+ \neq \mathrm{NaN}_0^+$ and similarly for all concrete and abstract NaN objects. Hence the Law of Identity does not hold in the standard. But the Law of Identity does hold in Classical and Boolean Logic so these logics cannot be used directly with NaN objects. We can, however, use these logics to describe meta logics in which the Law of Identity does not hold. The easiest way to do this is to operate on the bit patterns representing the NaN objects. Hence we cannot specify or verify a floating-point program entirely with abstract datatypes but are forced to specify the bit patterns of whichever NaN objects and zeros are in scope. Zeros are discussed below. Furthermore $x \neq x$ specifies a contradiction in any logic for which the Law of Identity holds. This is a problem because users do expect to use such logics, specification and verification methods, so they must accept that the standard is both non-logical and contradictory in parts. By contrast transreal arithmetic is both logical and consistent [13] [31] [19] [49] [47] [54] so it would avoid these consequences of NaN ordering.

Page 16 also says that an abstract infinity is produced by division by an abstract zero and that the concrete infinities have the ordering $-\infty < +\infty$. Hence $-\infty \neq +\infty$. The accompanying commentary is incoherent: "Infinity arithmetic shall be construed as the limiting case of real arithmetic with operands of arbitrarily large magnitude, when such limits exist." If "limit" means "boundary" then it cannot fail to exist in a finite model of arithmetic so "when such limits exist" is redundant. Alternatively if "limit" has its meaning in analysis then it is a category error confusing arithmetic with analysis. This commentary could be deleted or clarified by referring to the properties of transreal arithmetic.

Page 17 gives the rules for combining sign bits in multiplication and division but these lead independently to the devastating consequence just discussed. We have $0^- = 0^+$, $1/0^- = -\infty$, $1/0^+ = \infty$, but $-\infty \neq +\infty$, whence $0^- \neq 0^+$, which is a contradiction. Furthermore the Law of Identity can also be given as $x = y$ if and only if, for all functions, $f(z)$, it is the case that $f(x) = f(y)$. Now taking $f(z) = 1/z$ we immediately see that the Law of Identity does not hold in the standard. If 0^- is wanted then adopting the ordering $0^- < 0^+$, whence $0^- \neq 0^+$, would remove these problems.

Page 18 defines eight invalid operations. Five do not exist in transreal arithmetic, either because the operation does not exist or because it is valid. Preserving the standard's numbering, these are: (1) operations on signalling NaN objects; (2) subtraction of infinities; (3) multiplication of zero by infinity; (4) division of zero by zero or infinity by infinity; (8) some comparisons of NaN objects. In addition, we can define the remainder operation so that (5) $x \,\mathrm{REM}\, y$ is valid where $x = \pm\infty$ and where $y = 0$. This disposes of six of the eight invalid operations.

We do not consider the standard's appendix because it is out of scope of the standard.

4 2008 Standard

In this section we criticise the 2008 version of the IEEE standard [2]. All page numbers refer to that version. All of the criticisms in the previous section apply to this version but sometimes with some modification.

Page 7 says, "The mathematical structure underpinning the arithmetic in this standard is the extended reals, that is, the set of real numbers together with positive and negative infinity." This is false because the arithmetic in the standard also applies to NaN objects which cannot be embedded in the extended real number line because they are unordered. Furthermore the statement is unhelpful because there is no agreed arithmetic of the extended reals. It would, however, be possible to base a floating-point standard on the transreal number line shown in Figure 1.

The ambition to base the standard on the extended real number line, which has one missing cut, is an improvement on using the real number line, which has three missing cuts, but an even better proposal is to use the transreal number line which has no missing cuts [47].

Pages 29 and 30 give tables of 22 "functionally distinct useful" comparisons and negations, whereas the 1985 standard gives 26, and we expect $2^5 = 32$, including the empty combinations, or 30 excluding them.

Pages 41-45 define various mathematical functions, most of which fail on a NaN argument. Some of these definitions are inconsistent.

Consider the identity:

$$x^0 = e^{(\ln x)^0} = e^{0 \ln x}. \tag{1}$$

In real analysis this identity holds for all real $x \neq 0$. In transreal analysis it holds for all transreal x, [16] [26]. What is the case in the standard? Let us examine one case. Consider the positive concrete quiet NaN_i^+. Now $(\mathrm{NaN}_i^+)^0$ is computed as $\mathrm{pow}(\mathrm{NaN}_i^+, 0) = 1$. But:

$$\left(\mathrm{NaN}_i^+\right)^0 \to exp(0 * \ln(\mathrm{NaN}_i^+)) \to \mathrm{NaN}_i^+. \tag{2}$$

Here, in formula 2, we cannot use equality because NaN objects do not obey the Law of Identity so we have used a production arrow ($\to$) instead. We have also used an asterisk (*) to indicate multiplication. We now have the expressions "$(\mathrm{NaN}_i^+)^0 = 1$" and "$(\mathrm{NaN}_i^+)^0 \to \mathrm{NaN}_i^+$" but we cannot compare these expressions by any logical means, because the Law of Identity does not hold. However we can employ metalogical reasoning. If we are inclined toward logic, we may observe that the metalogical identity of 1 is different from the metalogical identity of NaN_i^+ or, if we are inclined toward computation, we may observe that the bit pattern for 1 is different from the bit pattern for NaN_i^+ in the non-sign bits, which is sufficient to show that they are not identical. Whatever metareasoning we employ, we come to the conclusion that "$(\mathrm{NaN}_i^+)^0 = 1$" and "$(\mathrm{NaN}_i^+)^0 \to \mathrm{NaN}_i^+$" are mutually inconsistent, which is to say that the standard's mathematical functions are inconsistent.

We leave it as an exercise for the diligent reader to show that those transreal functions, that correspond to the mathematical functions in the standard, are consistent.

The inconsistency of mathematical functions in the standard has arisen for a methodological reason. Consensus and formal vote are a good way to decide among valid alternatives in a standards' committee but they are not an effective alternative to mathematical proof.

Page 28 describes the ordering predicate *totalOrder.* This enforces $0^- < 0^+$, which agrees with our suggestion in the previous section, but it remains the case that $0^- = 0^+$, which further complicates the behaviour of abstract 0.

As noted in [4], the totalOrder predicate arranges that all negative concrete NaN_i^- order before all extended real numbers which, in turn, order before all positive concrete NaN_j^+. Similarly all positive concrete NaN_i^+ order after all extended real number which, in turn, order after all negative concrete NaN_j^-. Thus an abstract NaN_i, being the totality of its concrete representatives, NaN_i^- and NaN_i^+, orders both before and after all extended real numbers and orders before and after all abstract NaN objects, including itself. Which is to say that totalOrder does not order abstract NaN objects, making the name "totalOrder" anti-mnemonic. This problem could be resolved by respecting signs so that NaN_i^- and NaN_i^+ are different objects. Alternatively trans-floating-point arithmetic could be adopted.

We do not consider the standard's appendices because they lie outside the scope of the standard.

5 2019 Standard

All of the criticisms in the previous two sections apply to the 2019 version of the standard [3], sometimes with some modification. We note only that this version of the standard modified the ordering of concrete NaN objects, thereby creating a precedent for further adjusting this predicate so that it does order abstract NaN objects – if NaN objects are wanted.

6 Discussion

Our technological society depends very heavily on numerical computation. The IEEE standard for floating-point arithmetic is very widely used but we have proved that it is incoherent in all of its versions [1] [2] [3]. This is for three reasons.

Firstly the standard does not obey the Law of Identity so it is both non-logical and inconsistent, which prevents the specification and verification of floating-point programs at the level of abstract data but permits them at the level of bit patterns. These problems can be resolved by modifying the definitions of abstract zero and the abstract NaN objects.

The problem with abstract zero is that equal concrete zeros, $0^- = 0^+$, behave differently, $\frac{1}{0^-} \neq \frac{1}{0^+}$. The solution is to define that the concrete zeros are not equal, $0^- \neq 0^+$. A sensible way to do this is to define $0^- < 0^+$. The standard may then either take abstract 0 identical to concrete 0^+ so that $0 = 0^+ > 0^-$ or the standard could introduce a positive infinitesimal 0^+ such that $0^- < 0 < 0^+$. Alternatively the standard could abandon infinitesimals. Whatever near-zero concrete values the standard settles on, it can obtain any desired behaviour by defining suitable rounding modes.

There are two problems with the abstract NaN objects. Firstly $\mathrm{NaN}_i \neq \mathrm{NaN}_i$ directly contradicts the Law of Identity. This problem can be resolved by defining $\mathrm{NaN}_i = \mathrm{NaN}_i$ and $\mathrm{NaN}_i \neq \mathrm{NaN}_j$ when $i \neq j$, where the indexes (i, j) run over all concrete NaN objects. Consequently the redefined abstract NaN objects would respect sign. The second problem is that abstract NaN objects are not ordered by the totalOrder predicate but this would also be resolved by respecting the sign of NaN objects.

With these amendments in place, each abstract object has exactly one concrete representative so specification and verification of abstract data is bijective with specification and verification of concrete data. Hence it becomes possible to specify and verify floating-point programs at the level of abstract data.

Another way of proceeding would be to define a trans-floating-point standard and develop a migration path to it from the IEEE standard.

Secondly the four relational operators *less than, equal to, greater than, unordered* are not mutually exclusive, contradicting the commentary in the standard. Either the standard could correct the commentary or else mutually exclusive relational operators could be adopted, as in transreal arithmetic [4] [25].

Thirdly the standard has the methodological problem that the relevant IEEE committee defines the properties of mathematical functions by consensus and formal vote, without employing mathematical proof. This has lead to the inconsistent definition of mathematical functions.

Division by zero has been well defined since 1957, using calculi that do obey the Law of Identity [15]. Multiple calculi have been available throughout the life of the standard [14] and could have been used. We recommend transreal arithmetic as a foundation for computation because it is, so far as we know, the only division by zero calculus that employs all of the cuts of a line [47] and is, therefore, compatible with real arithmetic and all of its developments and applications, including physics. Compared to these advantages the current standard is a naked emperor.

The incoherence of the standard is a symptom of a wider problem which deserves to be taken seriously. When Dedekind defined the real numbers in terms of cuts of a line, he outlawed three cuts, which disallows division by zero. When these three cuts are included, we obtain transreal arithmetic [47], such that the previously outlawed cuts are the transreal numbers $-\infty, \Phi, +\infty$. Given that real and transreal arithmetic have an identical basis, it is questionable, if not perverse, to use real arithmetic, which is partial and fails on division by zero, when we could use transreal arithmetic, which is total and allows division by zero. Speaking bluntly: real arithmetic and all of its developments are guaranteed to fail because real arithmetic is partial – it fails on division by zero. By contrast, there is no guarantee of failure in transreal arithmetic and its developments. Choosing guaranteed failure is questionable, if not perverse, and is highly damaging to our technological society. We, or rather you, would do better to adopt transreal arithmetic.

The transreal versions of Newton's Laws of Motion and Gravitation hold for transreal numbers so division by zero is physically meaningful in these cases [22]. These laws generalise to many other physical systems, as the history of physics attests.

The history of transmathematics stands testament to the fact that it will take the academic community and industrial practice a long time to face up to the problems of division by zero faults in arithmetic and computer standards. In the mean time, various groups have obtained correct transreal computation by emulating it on conventional processors.

Trans-two's-complement arithmetic has been implemented in FPGA [5] and could therefore be implemented in IP cores by the manufacturers of FPGAs and as processing elements or floating-point units by the manufacturers of ASIC chips.

I know, from private communication, that trans-floating-point arithmetic has been implemented in Verilog, and could therefore be manufactured in FPGA IP cores and ASIC chips.

It is common practice, in the transmathematics community, to emulate transreal arithmetic in an IEEE 754 compliant floating-point processor as follows. Firstly all concrete $\mathrm{NaN}^{\pm}$ are mapped onto a single silent NaN^{+} that is treated as nullity. Compiler switches are selected to turn off the generation of signalling NaN objects. Concrete zeros, $0^{\pm}$, are folded onto abstract $0 = \mathrm{abs}(0^{\pm}) = 0^{+}$. The transreal comparisons less than, equal to, greater than are implemented. This gives fault-free transreal computation but at the twin costs of slower execution and missing out on the calculation of results to a tolerance that is twice as fine as the standard's arithmetic in the same number of bits. Nonetheless this is sufficient to implement transreal compilers, transreal programs, and simulators of transreal computers. Some groups go further and emulate transreal arithmetic directly using bit strings.

It has been claimed [67], without evidence, that transreal arithmetic plagiarised the IEEE standard for floating-point arithmetic . As there is no evidence, this is not a scientific claim so needs no rebuttal, but some scientists do believe the claim so addressing it performs a service to the academic community. In order to substantiate the claim, it would be necessary for the claimants to cite a paper on transmathematics, published no later than 2006, and cite which part of the 1985 standard [1] it plagiarises. The claimants would then need to address, at least, all of the material in Section 3 of the present paper to show that they have not cherry picked evidence. Furthermore they should explain away all of the recorded history of the independent development of transreal arithmetic sketched in Section 2. Finally they should give some account of how it is that an allegedly plagiarised system has more expressive power than the system it was allegedly copied from.

7 Conclusion

We have proved that all versions of the IEEE standard for floating-point arithmetic are non-logical, contradictory, and otherwise incoherent. We have shown how to resolve these problems in the standard's own terms and how to resolve them by basing floating-point arithmetic on transreal arithmetic, in which case trans-floating-point arithmetic would compute results to a tolerance twice as fine as the standard's arithmetics in the same number of bits. We argue that other computer arithmetics, applications, and mathematics itself would benefit from taking transreal arithmetic as a foundation and we note that some groups already obtain some or all of these advantages by emulating transreal arithmetic on standard processors.

We know of no evidence that transreal arithmetic plagiarised the IEEE standard but set out criteria for establishing that claim.

Acknowledgments

The author would like to thank Jan Aldert Bergstra, Walter Gomide, and Tiago S. dos Reis for reading an early draft of this paper and providing additional references. Thanks are also due to Stephen Leach for reading an early draft of this paper. All errors are the sole responsibility of the author.

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