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Transreal Explicit Construction of Universal Possible Worlds

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Abstract

In an earlier paper we supplied an indirect proof of the existence of universal possible worlds that have the topological property of being hypercyclic, which means they can access every world in sequences of worlds that approach arbitrarily closely to every possible world. That proof states that there are universal worlds but it does not exhibit such a world explicitly. We now explicitly construct two such universal worlds. A continuous universal world constructs possible worlds with transreal co-ordinates directly. A discrete world provides a binary hypercyclic vector which can be used to create transfloating-point co-ordinates that approximate transreal co-ordinates. We also discuss the philosophical implications of universal worlds for an omniscient observer and human science.

Introduction

In [8] we proposed a mathematical model for a *total semantics* and a *logical space*. For total semantics we understand a logical system which contains the classical values of truth and falsehood; a value of contradiction, inspired by paraconsistent logics; values which correspond to degrees of truth and degrees of falsehood, inspired by fuzzy logics; and a gap value which corresponds to the indeterminacy, that is, a value which does not contain information about the truth or falsehood of a sentence. Furthermore, this system has the logical connectives of negation, disjunction and conjunction acting appropriately in the values just cited; that is, having the expected action on the classical, dialetheaic, fuzzy, and gap values. The idea of logical space, in turn, is inspired by Wittgenstein's conception that the world's logical form is given by a "configuration of objects." Thus, just as physical objects are arranged in a physical space, objects which logically make up the world are situated in a "logical space" [7]. We have established logical space as a well-defined mathematical structure like a vector space, with the possible worlds being transvectors and the communication between them being transvector transformations.

We chose the set of transreal numbers to translate the set of semantic values. We defined the set of semantic values as $\mathbb{R}^T$. We defined the space of the sequences of transreal numbers, $(\mathbb{R}^T)^{\mathbb{N}}$, as the set of all possible worlds and used the topological and transvectorial structure of this space to model several logical concepts. A topological space is a set where it makes sense to speak about neighbourhood, proximity and convergence. This allowed us to give a mathematical sense to the idea that a possible world is close to another one and to the idea that a succession of possible worlds converges to a determinate one. In this way, we were able to give a mathematical definition to the notion of an *accessibility relation,* motivated in modal logic, that allows the motion from one possible world to another. We proved that there is a *universal world,* that is, a possible world which can access any other by approximating it.

That proof was not explicit. That is, it proved that infinitely many universal worlds exist but it did not exhibit such a world explicitly. In the present paper we give an explicit construction of two, different, universal possible worlds: the first is a continuous world and the second is a discrete world. This presentation, though pedagogically useful, does reverse the historical order in which these constructions were obtained.

Before we begin, let us clarify notation. In Computer Science it is usual to define that the natural numbers are all of the positive integers and zero, whereas in Mathematics it is usual to define that the natural numbers are just the positive integers, excluding zero. Here we follow the mathematical convention and define that the natural numbers exclude zero and are given by $\mathbb{N} = \{1, 2, 3, \dots\}$.

1 Preliminary: Possible Worlds and the Existence of a Universal World

In order to understand the transreal model of the space of possible worlds, in this section, we summarise the content of [8].

The method of proof was to establish the space of all possible worlds as a geometrical space and to establish certain algebraic and topological properties of that space. Each axis of the geometrical space was labelled with a unique atomic proposition so that a co-ordinate, on a labelled axis, is the degree to which the labelled proposition is True, False or Gap. Thus points in this space are arrangements of semantic values of the atomic propositions. In other words, points in this space are possible worlds.

The work started by considering what set should be used as semantic values. We chose the set of transreal numbers.

The set of transreal numbers, $\mathbb{R}^T$, [4] [14] is made up of the real numbers, together with three, definite, non-finite numbers: negative infinity, positive infinity, and nullity, $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$. In Figure 1 the real numbers are shown as a continuous line of some finite length in the figure. The axis is scaled to allow all real numbers to be laid out in the figure. Positive infinity, ∞, lies to the right of the real-number line, but after a space. This space is a necessary and essential property of the transreal numbers [3] [13] [12]. Similarly negative infinity, $-\infty$, lies to the left of the real-number line, after a space. Nullity, Φ, lies off the real number line. All of the real numbers and both positive and negative infinity are ordered so that negative infinity is the smallest of these numbers and positive infinity is the largest of them. Nullity is not ordered, it is neither small nor large, nor any size in between. Its size is nullity.

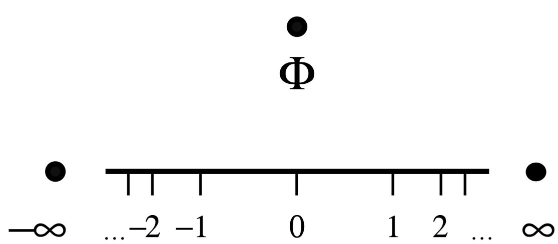

Figure 1: Transreal Number Line

The set of transreal numbers is a metric space with the following metric: $d : \mathbb{R}^T \times \mathbb{R}^T \to \mathbb{R}$,

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 2, & \text{if } x = \Phi \text{ or } y = \Phi \\ |\varphi(x) - \varphi(y)|, & \text{otherwise} \end{cases}, \tag{1}$$

where φ is the homeomorphism $\varphi : [-\infty, \infty] \to [-1, 1]$,

$$\varphi(x) = \begin{cases} -1 & , \text{ if } x = -\infty \\ \dfrac{x}{1+|x|} & , \text{ if } x \in \mathbb{R} \\ 1 & , \text{ if } x = \infty \end{cases} .$$

With this metric, $\mathbb{R}^T$ is a Hausdorff, disconnected, separable and compact space [12].

Each element of $\mathbb{R}^T$ is called a *semantic value*. Hence $\mathbb{R}^T$ is *the set of semantic values*. The connective *negation* is given by

$$\begin{array}{rccc} \neg : & \mathbb{R}^T & \longrightarrow & \mathbb{R}^T \\ & x & \longmapsto & \neg x = -x \end{array} .$$

The connective *disjunction* is given by

$$\begin{array}{rccl} \vee : & \mathbb{R}^T \times \mathbb{R}^T & \longrightarrow & \mathbb{R}^T \\ & (x, y) & \longmapsto & x \vee y = \begin{cases} \Phi & , \text{ if } x = \Phi \text{ or } y = \Phi \\ \max\{x, y\} & , \text{ otherwise} \end{cases} \end{array}$$

and the connective *conjunction* is given by

$$\begin{array}{rccl} \wedge : & \mathbb{R}^T \times \mathbb{R}^T & \longrightarrow & \mathbb{R}^T \\ & (x, y) & \longmapsto & x \wedge y = \begin{cases} \Phi & , \text{ if } x = \Phi \text{ or } y = \Phi \\ \min\{x, y\} & , \text{ otherwise} \end{cases} \end{array} .$$

In [8] we generalised the notion of Boolean logic to trans-Boolean logic. A trans-Boolean algebra is a structure $(X, \neg, \vee, \wedge, \bot, \top)$, where X is a set, $\bot, \top \in X$, $\neg$ is a function from X to X, and $\vee$ and $\wedge$ are functions from $X \times X$ to X such that the following properties are satisfied: (i) existence of an identity element, (ii) commutativity, (iii) associativity and (iv) distributivity. Thus, for all $x, y, z \in X$: (i) $x \vee \bot = x$ and $x \wedge \top = x$; (ii) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$; (iii) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$; (iv) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

We proved that the transreal numbers do model classical, fuzzy and a particular paraconsistent logic by establishing homomorphisms between these logics and trans-Boolean logic ([8], Theorem 2.5). With the logical connectives defined above, it follows that: negative infinity models the classical truth value False and positive infinity models the classical truth value True; the set of real numbers models dialeathic values that have degrees of both falsehood and truthfulness [6] [9] [10] [11]: negative values are more False than True, positive values are more True than False, zero is equally False and True; and Nullity models gap values that are neither False nor True and which, more generally, have no degree of falsehood or truthfulness [16] [17]. Thus one can model the semantic values of many logics.

The next step in [8] was to define a geometrical model for the space of all possible worlds and an accessibility relation between them. Thus, intuitively, a possible world is a binding of a proposition to its semantic values. That is, at a given possible world, each atomic proposition takes on a semantic value in $\mathbb{R}^T$. We assumed, as usual, that the set of atomic propositions is a countable set. Hence the set of atomic propositions can be written in the form $\{P_i;\ i \in \mathbb{N}\} = \{P_1, P_2, P_3, \ \dots\}$, where $P_i \neq P_j$ whenever $i \neq j$. Thus we can interpret a possible world as a function from $\{P_1, P_2, P_3, \dots\}$ to $\mathbb{R}^T$. But this forms a sequence of elements from $\mathbb{R}^T$. So, denoting the set of the sequences of elements from $\mathbb{R}^T$ by $(\mathbb{R}^T)^{\mathbb{N}}$, we adopted the following definition. Each element of $(\mathbb{R}^T)^{\mathbb{N}}$ is called a *possible world*. Hence $(\mathbb{R}^T)^{\mathbb{N}}$ is *the set of all possible worlds*. In this way each possible world is a point in the space $(\mathbb{R}^T)^{\mathbb{N}}$. Given a possible world $w = (w_i)_{i\in\mathbb{N}} \in (\mathbb{R}^T)^{\mathbb{N}}$, for each $i \in \mathbb{N}$, w_i corresponds to the semantic value of P_i in w.

We proposed a mathematical object that plays the role of an *accessibility relation* from a possible world w to a possible world u. The existence of a continuous linear transformation T on $(\mathbb{R}^T)^{\mathbb{N}}$, such that $T(w) = u$ is a relation with the desired characteristics. But to manifest this definition, it was necessary to define vector operations on $(\mathbb{R}^T)^{\mathbb{N}}$. Let $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, where $w = (w_j)_{j\in\mathbb{N}}$ and $u = (u_j)_{j\in\mathbb{N}}$, and $x \in \mathbb{R}^T$. We defined $w+u := (w_j+u_j)_{j\in\mathbb{N}}$ and $xw := (xw_j)_{j\in\mathbb{N}}$ and denoted $(0, 0, 0, \dots) \in (\mathbb{R}^T)^{\mathbb{N}}$ simply by 0. With these operations, $(\mathbb{R}^T)^{\mathbb{N}}$ is a transvector space. A non empty set, V, is called a *transvector space on* $\mathbb{R}^T$ if and only if there are two operations $+ : V \times V \longrightarrow V$ and $\cdot : \mathbb{R}^T \times V \longrightarrow V$ (named, respectively, *addition* and *scalar multiplication*), such that the following properties are satisfied: additive commutativity, additive associativity, scalar multiplicative associativity, additive identity and scalar multiplicative identity. Which are, respectively, for any $w, u, v \in V$ and $x, y \in \mathbb{R}^T$: (i) $w + u = u + w$; (ii) $w+(u+v) = (w+u)+v$; (iii) $x\cdot(y\cdot w) = (xy)\cdot w$; (iv) there is $o \in V$ such that $o + w = w$ and (v) $1 \cdot w = w$. The elements of V are called *transvectors*. Further $x \cdot w$ is customarily denoted as $w \cdot x$, xw or wx and o as 0 on $\mathbb{R}^T$ [8].

We defined a translinear transformation as follows. Let V and W be transvector spaces on $\mathbb{R}^T$. We say that $T : V \to W$ is a *translinear transformation* on V if and only if for all $w, u \in V$ and $x \in \mathbb{R}^T$: (i) $T(w+u) = T(w)+T(u)$ and (ii) $T(xw) = xT(w)$ [8].

Now, given two arbitrary possible worlds $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, $T : (\mathbb{R}^T)^{\mathbb{N}} \longrightarrow (\mathbb{R}^T)^{\mathbb{N}}$ is called a *communication from w to u* if and only if T is a continuous, translinear transformation and satisfies: (i) every constant sequence is a fixed point of T, in other words, for each $v = (v_i)_{i\in\mathbb{N}} \in (\mathbb{R}^T)^{\mathbb{N}}$ such that $v_i = v_1$ for all $i \in \mathbb{N}$, $T(v) = v$ and (ii) $T(w) = u$.

Given two arbitrary possible worlds $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, it is said that wRu if and only if there is a communication from w to u. We call the relation R an *accessibility relation*. In other words, w *accesses* u or u *is accessible from* w if and only if wRu. The accessibility relation is reflexive and transitive. See [8].

Finally, we proved the existence of worlds which approximate any worlds. The proof used the concept of *hypercyclicity* [15] from functional analysis. We extended the topological notion of hypercyclicity so that it holds in the logical space. When a vector is operated on by a certain kind of operator, it generates new vectors in a structure called an orbit. The elements from the orbit of a hypercyclic vector lie arbitrarily closely to any element in the space and sequences of elements can be chosen, from the orbit, so that they converge, arbitrarily closely, to any element in the space. We used the backward shift operator to generate an orbit of possible worlds from a single, hypercyclic, possible world. The backward shift operator shuffles all of its co-ordinate values down one place so that the first co-ordinate value drops off the beginning of the vector, in a process that is exactly like running Hilbert's hotel paradox backwards. Some sequences of possible worlds then converge arbitrarily closely to any particular possible world and there are so many sequences that every possible world is approached in this way. The proof shows that there are infinitely many hypercyclic, or universal, worlds and that these are spread with infinite density throughout the space of possible worlds.

We showed that $(\mathbb{R}^T)^{\mathbb{N}}$ is a metrizable space with the metric D defined as follows. For each $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, denote $w = (w_j)_{j\in\mathbb{N}}$ and $u = (u_j)_{j\in\mathbb{N}}$, and let $D : (\mathbb{R}^T)^{\mathbb{N}} \times (\mathbb{R}^T)^{\mathbb{N}} \to \mathbb{R}$ be given as

$$D(w,u) = \sup_{j\in\mathbb{N}} \left\{ \frac{d(w_j, u_j)}{j} \right\}, \tag{2}$$

where d is defined in (1). We showed that $(\mathbb{R}^T)^{\mathbb{N}}$ is a metric, complete, separable and compact space ([8], Remark 3.7 and Corollary 4.2).

In a metric space we can speak of distance between elements and we can speak of neighbourhoods, proximity and convergence. We can give an exact sense to the notion of "being close to." According to our model of logical space, possible worlds are points in the metric space $(\mathbb{R}^T)^{\mathbb{N}}$. This allows us to speak of proximity and convergence with respect to possible worlds. That is, there is an exact meaning to "a possible world is close to another" and to "a succession of possible worlds converges to a particular world."

Let X be a topological space. A continuous operator, T on X, is said to be *hypercyclic* if and only if there is an $x \in X$ such that $\mathrm{orb}(x, T)$ is dense in X. In this case x is called an *hypercyclic element* of T. Given a set X and a function $f : X \longrightarrow X$, the *iterates* of f are defined as $f^0 = \mathrm{Id}_X,\ f^1 = f,\ f^2 = f \circ f,\ f^3 = f \circ f^2,\ \ldots$; where Id_X the identity function on X. Also, for each $x \in X$, the *orbit of x related to f* is defined as $\mathrm{orb}(x, f) := \{x, f(x), f^2(x), \ldots\}$.

The fact that $\mathrm{orb}(x, T)$ is dense in X means, in the model of logical space, that a hypercyclic world generates a sequence of worlds such that every world is approached, arbitrarily closely, by worlds in that sequence.

Let $B : (\mathbb{R}^T)^{\mathbb{N}} \longrightarrow (\mathbb{R}^T)^{\mathbb{N}},\quad B(w_1, w_2, w_3, \ldots) = (w_2, w_3, w_4, \ldots)$. The operator B is called a *backward shift*. We showed that B is an hypercyclic operator on $(\mathbb{R}^T)^{\mathbb{N}}$ ([8], Theorem 3.13). Being hypercyclic means that there is a possible world, w, such that, given any possible world u, there is a possible world, v, which is metrically as close to u as one can want, such that w accesses v. In other words, w accesses any possible world "by metrical approximation." That proof of the existence of universal worlds is indirect, it is not a constructive proof. That is, the proof stated that there are universal worlds but did not exhibit such a world explicitly.

2 Explicit Construction of a Continuous Transreal Universal Possible World

Rolewicz has explicitly constructed a hypercyclic vector in certain spaces of sequences of real numbers [15]. He starts with an arbitrary, dense sequence in the underlying space. Here we adapt his construction to the space of possible worlds - which is the space of all sequences of transreal numbers - with the difference that we start with a specific, explicitly constructed, dense sequence in our space so that our construction is completely explicit. Thus we show an explicit construction of a universal world.

2.1 An Outline of the Construction

Saying that a transvector w is hypercyclic, with respect to the backward shift operator in the space of possible worlds, means that the orbit of w, with respect to B, denoted as $\mathrm{orb}(w, B)$, is dense in $(\mathbb{R}^T)^{\mathbb{N}}$. Saying that $\mathrm{orb}(w, B)$ is dense in $(\mathbb{R}^T)^{\mathbb{N}}$ means that we find elements from $\mathrm{orb}(w, B)$ arbitrarily close to any element of $(\mathbb{R}^T)^{\mathbb{N}}$. That is, saying that $\mathrm{orb}(w, B)$ is dense in $(\mathbb{R}^T)^{\mathbb{N}}$ means that given any element u from $(\mathbb{R}^T)^{\mathbb{N}}$, there exists some element v from $\mathrm{orb}(w, B)$ such that v is arbitrarily close to u. Saying that there exists v arbitrarily close to u means that given any positive ε, there exists v such that the distance between v and u is less than ε. That is, saying that there exists v arbitrarily close to u means for every positive ε there exists v such that $D(v, u) < \varepsilon$.

However, denoting $v = (v_1, v_2, \dots)$ and $u = (u_1, u_2, \dots)$, we have that $D(v, u) = \sup\left\{\frac{d(v_1, u_1)}{1}, \frac{d(v_2, u_2)}{2}, \dots\right\}$. Thus, in order that $D(v, u) < \varepsilon$ it suffices that $\frac{d(v_i, u_i)}{i} < \varepsilon$ for all $i \in \mathbb{N}$. But, the function distance d is bounded by 2, that is, $d(\alpha, \beta) \leq 2$ for all $\alpha, \beta \in \mathbb{R}^T$ whence $d(v_i, u_i) \leq 2$ for all $i \in \mathbb{N}$. Hence, $\frac{d(v_i, u_i)}{i} < \varepsilon$ for all i sufficiently large. In other words, there is $n \in \mathbb{N}$ such that $\frac{d(v_i, u_i)}{i} < \varepsilon$ for all $i > n$. Thus, in order for $\frac{d(v_i, u_i)}{i} < \varepsilon$ for every $i \in \mathbb{N}$, we need to be concerned only with the first n co-ordinates of v and u. In short, in order that $D(v, u) < \varepsilon$, we need only that finitely many co-ordinates of v are close to the respective co-ordinates of u.

In this way, saying that given any element u from $(\mathbb{R}^T)^{\mathbb{N}}$, there exists some element v from $\mathrm{orb}(w, B)$ such that v is arbitrarily close to u means that given u', a finite sequence of transreal numbers, there exists a finite sequence v', which is part of some element from $\mathrm{orb}(w, B)$, such that the co-ordinates of v' are arbitrarily close to the co-ordinates of u'.

Since u' can be any finite sequence of transreals, we need v' to be chosen from a dense set of finite sequences. However, v' needs to be a finite part of some element from $\mathrm{orb}(w, B)$ and $\mathrm{orb}(w, B)$ is a countable set. But $\mathbb{Q}$ is a countable set and $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus, we may choose v' from the set of all finite sequences of rational numbers and nullity. Denote $\mathbb{Q}_\Phi := \mathbb{Q} \cup \{\Phi\}$. In this way, we need that every finite sequence of elements from $\mathbb{Q}_\Phi$ is part of some element from $\mathrm{orb}(w, B)$. However, all elements from $\mathrm{orb}(w, B)$ are obtained by recursive applications of the backward shift operator to the transvector w. That is, every element from $\mathrm{orb}(w, B)$ is the transvector w, except for a finite initial piece of it. Thus, in order for any finite sequence of elements from $\mathbb{Q}_\Phi$ to be part of some element from $\mathrm{orb}(w, B)$, we need that every finite sequence of elements from $\mathbb{Q}_\Phi$ is part of w. Therefore, w must contain, in its formation, all finite sequences of elements from $\mathbb{Q}_\Phi$.

Eureka! It suffices for w to be the juxtaposition of all finite sequences of elements from $\mathbb{Q}_\Phi$. So to construct such a hypercyclic transvector explicitly, we need to explicitly enumerate all finite sequences of rational numbers. After we have this enumeration, we put its elements side by side. For example, let us say that 0; -1; $(1, \Phi)$; $(0, \Phi, 1)$; $(-1, 1)$ are the first five elements of this enumeration. In this hypothetical example, the first nine elements of w would be $0, -1, 1, \Phi, 0, \Phi, 1, -1, 1$.

If in 0; -1; $(1, \Phi)$; $(0, \Phi, 1)$; $(-1, 1)$; $\ldots$ all finite sequences of elements from $\mathbb{Q}_\Phi$ appear then any finite sequence of elements from $\mathbb{Q}_\Phi$ is a part of w. Assuming that w has been constructed in the way mentioned above, given an arbitrary sequence of elements from $\mathbb{Q}_\Phi$ with n elements, we are certain that we can apply the backward shift operator to w a sufficient number of times so that the first n terms of the result of this recursive application is the sequence given. That is, given any sequence $(v_1, \ldots, v_n)$ of elements from $\mathbb{Q}_\Phi$, there exists $s \in \mathbb{N} \cup \{0\}$ such that the first n co-ordinates of $B^s(w)$ are $v_1, \ldots, v_n$.

Thus when u', an arbitrary sequence of transreal numbers with n co-ordinates, is given to us, we will take a sequence, v', of elements from $\mathbb{Q}_\Phi$ with n co-ordinates so that the k-th co-ordinate of v' is arbitrarily close to the k-th co-ordinate of u' (this is possible because $\mathbb{Q}_\Phi$ is dense in $\mathbb{R}^T$). Next we will take $s \in \mathbb{N} \cup \{0\}$ such that the first n co-ordinates of $B^s(w)$ are the n co-ordinates of v'.

When an arbitrary element u from $(\mathbb{R}^T)^\mathbb{N}$ and an arbitrary positive ε is given to us, we will choose:

- $n \in \mathbb{N}$ such that $\dfrac{2}{i} < \varepsilon$ for all i greater than n,
- a sequence v' of elements from $\mathbb{Q}_\Phi$ with n co-ordinates such that $d(v_i, u_i) < \varepsilon$ for all i less than n and
- $s \in \mathbb{N} \cup \{0\}$ such that the first n co-ordinates of $B^s(w)$ are the n co-ordinates of v'.

Hence we will have $D(B^s(w), u) < \varepsilon$. Since ε is arbitrary among the positive real numbers, we will have found a $B^s(w)$ arbitrarily close to u. Since u is an arbitrary element of $(\mathbb{R}^T)^\mathbb{N}$ and $B^s(w)$ belongs to $\mathrm{orb}(w, B)$, we have it that $\mathrm{orb}(w, B)$ is dense in $(\mathbb{R}^T)^\mathbb{N}$. That is, w is hypercyclic in $(\mathbb{R}^T)^\mathbb{N}$.

Now, let us go back and see how to construct the transvector, w, as mentioned above. Let us look at how to make a juxtaposition of all finite sequences of elements from $\mathbb{Q}_\Phi$. For this, let us look at how to make an explicit enumeration of all finite sequences of elements from $\mathbb{Q}_\Phi$.

Imagine that we have an enumeration of all elements from $\mathbb{Q}_\Phi$, an enumeration of all pairs of elements from $\mathbb{Q}_\Phi$, an enumeration of all triples of elements from $\mathbb{Q}_\Phi$, and so on.

$$\begin{array}{ccccccc}
\mathbb{Q}_\Phi: & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
\mathbb{Q}_\Phi^2: & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\
\mathbb{Q}_\Phi^3: & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\
\mathbb{Q}_\Phi^4: & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

In the table above, for all $i \in \mathbb{N}$: a_{1i} is an element from $\mathbb{Q}_\Phi$; a_{2i} is an ordered pair of elements from $\mathbb{Q}_\Phi$; a_{3i} is an ordered triple of elements from $\mathbb{Q}_\Phi$; and so on. So we can use a diagonal argument to obtain a unique enumeration of all finite sequences of elements from $\mathbb{Q}_\Phi$.

$$\begin{array}{ccccccc}
\mathbb{Q}_\Phi: & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
\mathbb{Q}_\Phi^2: & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\
\mathbb{Q}_\Phi^3: & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\
\mathbb{Q}_\Phi^4: & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Having this enumeration, we simply ignore the finite sequences and look at each co-ordinate, in the order it appears, as the co-ordinate of a single infinite transvector.

To conclude our construction we need an enumeration, for each n, of the set of sequences with n elements. We will start by enumerating the rational numbers using the Calkin-Wilf tree. After enumerating the rationals, we will use the same diagonal argument to enumerate $\mathbb{Q}_\Phi^n$.

2.2 The Construction

Let $b : \mathbb{N} \to \mathbb{N}$ be defined recursively as

$$b(1) = 1,\ b(2k) = b(k) \text{ and } b(2k+1) = b(k) + b(k+1) \text{ for all } k \in \mathbb{N}.$$

Let $r : \mathbb{N} \to \mathbb{Q}^+$ be defined as

$$r(i) = \frac{b(i)}{b(i+1)} \text{ for all } i \in \mathbb{N}.$$

Notice that r is a bijection. This explicit enumeration of positive rational numbers was proposed by Neil Calkin and Herbert Wilf [5].

Let $t : \mathbb{Z} \to \mathbb{Q}$ be defined as

$$t(0) = 0,\ t(i) = r(i) \text{ and } t(-i) = -r(i) \text{ for all } i \in \mathbb{N}.$$

Notice that t is a bijection.

Let $h : \mathbb{N} \to \mathbb{Z}$ be defined as

$$h(1) = 0 \text{ and } h(i+1) = (-1)^i \left\lfloor \frac{i+1}{2} \right\rfloor \text{ for all } i \in \mathbb{N}.$$

Notice that h is a bijection.

Denote $\mathbb{Q}_\Phi := \mathbb{Q} \cup \{\Phi\}$. Let $f : \mathbb{N} \to \mathbb{Q}_\Phi$ be defined as

$$f(1) = \Phi \text{ and } f(i+1) = t(h(i)) \text{ for all } i \in \mathbb{N}.$$

Notice that f is a bijection.

For each $i \in \mathbb{N}$, let n_i be the exponent of the number 2 in the prime factorisation of i and denote $n_i + 1$ as $p(i)$ and denote $\frac{\frac{i}{2^{n_i}} + 1}{2}$ as $q(i)$, that is, $p(i) = n_i + 1$ and $q(i) = \frac{\frac{i}{2^{n_i}} + 1}{2}$. Let $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be defined as

$$g(i) = (p(i), q(i)) \text{ for all } i \in \mathbb{N}.$$

Notice that g is a bijection.

Let $f_1 : \mathbb{N} \to \mathbb{Q}_\Phi$ be defined as

$$f_1 = f;$$

let $f_2 : \mathbb{N} \to \mathbb{Q}_\Phi^2$ be defined as

$$f_2 = (f \circ p, f \circ q);$$

and, for each $k \in \mathbb{N} \setminus \{1, 2\}$, let $f_k : \mathbb{N} \to \mathbb{Q}_\Phi^k$ be defined as

$$f_k = (f \circ p \circ q^0, \ldots, f \circ p \circ q^{k-2}, f \circ q^{k-1}).$$

Notice that f_k is a bijection for all $k \in \mathbb{N}$.

Let $a : \mathbb{N} \to \bigcup_{k \in \mathbb{N}} \mathbb{Q}_\Phi^k$ be defined as

$$a(m) = f_{p(m)}(q(m)) \text{ for all } m \in \mathbb{N}.$$

Notice that a is a bijection.

Now, let us define the hypercyclic transvector $w \in (\mathbb{R}^T)^{\mathbb{N}}$. For each $k \in \mathbb{N}$ and each $i \in \mathbb{N}$, define $\pi_{k,i} : \mathbb{Q}_\Phi^k \to \mathbb{Q}_\Phi$ where, for each $x = (x_1, \ldots, x_k) \in \mathbb{Q}_\Phi^k$,

$\pi_{k,i}(x) = x_i$ when $i \leq k$ and $\pi_{k,i}(x) = 0$ when $i > k$. Define $\pi_i : \bigcup_{k\in\mathbb{N}} \mathbb{Q}_\Phi^k \to \mathbb{Q}_\Phi$ where, for each $x \in \bigcup_{k\in\mathbb{N}} \mathbb{Q}_\Phi^k$, $\pi_i(x) = \pi_{k,i}(x)$ when $x \in \mathbb{Q}_\Phi^k$. We denote $s_0 := 0$ and, for each $m \in \mathbb{N}$, we denote $s_m = \sum_{k=1}^m p(k)$. We define $w \in (\mathbb{R}^T)^\mathbb{N}$ in the following way: $w_i := \pi_i(a(1))$ when $i \leq p(1)$ and, for each $m \in \mathbb{N}$, $w_i := \pi_{i-s_m}(a(m+1))$ when $s_m < i \leq s_{m+1}$.

Theorem 2.1 *The element w, as defined above, is hypercyclic with respect to the backward shift in the space $(\mathbb{R}^T)^\mathbb{N}$.*

Proof. Let $u \in (\mathbb{R}^T)^\mathbb{N}$ be arbitrary. Given $\varepsilon > 0$ arbitrary, take $n \in \mathbb{N}$ such that $n > \frac{2}{\varepsilon}$ and $v = (v_1, \ldots, v_n) \in \mathbb{Q}_\Phi^n$ such that

$$d(v_i, u_i) < \varepsilon \text{ for all } i \in \{1, \ldots, n\}. \tag{3}$$

As $v \in \bigcup_{k\in\mathbb{N}} \mathbb{Q}_\Phi^k$, there is $m \in \mathbb{N}$ such that $a(m) = v$ whence $f_{p(m)}(q(m)) = v$ and $p(m) = n$. Notice that

$$\begin{array}{rcl} B^{s_{m-1}}w & = & (\, w_{s_{m-1}+1}, \;\ldots, \; w_{s_{m-1}+p(m)}, \; w_{s_{m-1}+p(m)+1}, \;\ldots) \\ & = & (\pi_1(a(m)), \;\ldots, \; \pi_{p(m)}(a(m)), \; w_{s_{m-1}+p(m)+1}, \;\ldots) \\ & = & (\pi_1(v), \;\ldots, \; \pi_{p(m)}(v), \; w_{s_{m-1}+p(m)+1}, \;\ldots) \\ & = & (v_1, \;\ldots, \; v_{p(m)}, \; w_{s_{m-1}+p(m)+1}, \;\ldots) \\ & = & (v_1, \;\ldots, \; v_n, \; w_{s_{m-1}+n+1}, \;\ldots) \end{array}$$

whence, by (2),

$$\begin{aligned} & D\left(B^{s_{m-1}}w, u\right) \\ = & \sup\left\{\frac{d(w_{s_{m-1}+1}, u_1)}{1}, \ldots, \frac{d(w_{s_{m-1}+p(m)}, u_{p(m)})}{p(m)}, \frac{d(w_{s_{m-1}+p(m)+1}, u_{p(m)+1})}{p(m)+1}, \ldots\right\} \\ = & \sup\left\{\frac{d(v_1, u_1)}{1}, \ldots, \frac{d(v_n, u_n)}{n}, \frac{d(w_{s_{m-1}+n+1}, u_{n+1})}{n+1}, \ldots\right\}. \end{aligned}$$

But, for all $\alpha, \beta \in \mathbb{R}^T$, by (1), $d(\alpha, \beta) \leq 2$ whence

$$\frac{d(w_{s_{m-1}+i}, u_i)}{i} < \frac{2}{n} < \varepsilon \text{ for all } i \in \{n+1, n+2, \ldots\}. \tag{4}$$

Thus, by (3) and (4), $D\left(B^{s_{m-1}}w, u\right) < \varepsilon$. As u was taken in $(\mathbb{R}^T)^\mathbb{N}$ arbitrarily, it follows that $\mathrm{orb}(w, B)$ is dense in $(\mathbb{R}^T)^\mathbb{N}$ whence w is hypercyclic with respect to B. ■

This construction of the universal world, w, is so explicit that we can determine the exact transreal number which is the i-th co-ordinate of w for all $i \in \mathbb{N}$. For example, let us determinate w_{20}. Notice that $p(1) = 1$, $p(2) = 2$, $p(3) = 1$, $p(4) = 3$, $p(5) = 1$, $p(6) = 2$, $p(7) = 1$, $p(8) = 4$, $p(9) = 1$, $p(10) = 2$, $p(11) = 1$

and $p(12) = 3$ whence $s_{11} = \sum_{k=1}^{11} p(k) = 19 < 20 \leq 22 = \sum_{k=1}^{12} p(k) = s_{11+1}$. Hence, $w_{20} = \pi_{20-s_{11}}(a(11+1)) = \pi_{20-19}(a(12)) = \pi_1(a(12))$. But $a(12) = f_{p(12)}(q(12)) = f_3(q(12))$ and $q(12) = \dfrac{\frac{12}{2^{p(12)-1}}+1}{2} = \dfrac{\frac{12}{2^{3-1}}+1}{2} = 2$ whence $a(12) = f_3(2)$. But $f_3(2) = ((f \circ p)(2), (f \circ p \circ q)(2), (f \circ q \circ q)(2)) = (f(p(2)), f(p(q(2))), f(q(q(2))))$ and $q(2) = 1$ and $q(1) = 1$ whence $f_3(2) = (f(p(2)), f(p(1)), f(q(1))) = (f(2), f(1), f(1))$. But $f(1) = \Phi$ and $f(2) = t(h(1)) = t(0) = 0$ whence $f_3(2) = (0, \Phi, \Phi)$. Thus, $a(12) = (0, \Phi, \Phi)$ whence $\pi_1(a(12)) = 0$. Therefore, $w_{20} = 0$.

3 Explicit Construction of a Discrete Transreal Universal Possible World

In this section we explicitly construct a vector with a dense orbit, with respect to the backward shift operator, in the Cantor Space $\{0,1\}^{\mathbb{N}}$. This vector is one of our possible worlds, albeit one whose every co-ordinate is zero or one, not a general transreal number.

The vector is a particular, indefinitely long, binary sequence which, by construction, enumerates all finitely long, binary sequences. Computer Science makes very heavy, almost universal, use of binary sequences. Here we use it to enumerate trans-floating-point numbers [1] [3] but, more generally, we could use it to enumerate many, if not all, of the abstract objects Computer Science considers.

A *Cantor Space* is a topological space that is homeomorphic to the Cantor set. A very simple Cantor space is $\{0,1\}^{\mathbb{N}}$, the countably infinite, topological product of the discrete space $\{0,1\}$. In other words, we take $\{0,1\}$ with the discrete topology, that is, the open sets of the space $\{0,1\}$ are: $\emptyset$, $\{0\}$, $\{1\}$ and $\{0,1\}$. And $\{0,1\}^{\mathbb{N}} = \prod_{j\in\mathbb{N}}\{0,1\} = \{0,1\} \times \{0,1\} \times \cdots$ is endowed with the product topology. Recall that, in a product topology, $U \subset \{0,1\}^{\mathbb{N}}$ is open if and only if $U = \prod_{j\in\mathbb{N}} U_j = U_1 \times U_2 \times \cdots$, where U_j is open on $\{0,1\}$ for all $j \in \mathbb{N}$ and $U_j = \{0,1\}$, except for finite many indexes j.

Let $B : \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^{\mathbb{N}}$ be the backward shift operator, as described above. Whence, $B(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$ for every $(x_1, x_2, x_3, \dots) \in \{0,1\}^{\mathbb{N}}$. Notice that B is continuous.

We now explicitly construct an element with a dense orbit, with respect to B in $\{0,1\}^{\mathbb{N}}$. Firstly, for each $n \in \mathbb{N}$, denote $s_n = 2^0 \times 0 + 2^1 \times 1 + 2^2 \times 2 + \cdots + 2^{n-1} \times (n-1)$. Secondly let $v = (v_1, v_2, v_3, \dots) \in \{0,1\}^{\mathbb{N}}$ be defined in the following way: for each $n \in \mathbb{N}$ and each $k \in \{0, 1, 2, \dots, 2^n-3, 2^n-2, 2^n-1\}$,

$$v_{s_n+kn+1} v_{s_n+kn+2} \cdots v_{s_n+kn+(n-1)} v_{s_n+kn+n}$$

is the binary representation of k in n digits.

To facilitate understanding of the element v we give some of its initial co-ordinates.

- For $n = 1$ we have $s_1 = 2^0 \times 0 = 0$ and k sweeping through $\{0, \ldots, 2^1 - 1\} = \{0, 1\}$.
  - For $k = 0$, $v_{s_1+0\times1+1} = v_1$ is the binary representation of $k = 0$ in $n = 1$ digit whence $v_1 = 0$.
  - For $k = 1$, $v_{s_1+1\times1+1} = v_2$ is the binary representation of $k = 1$ in $n = 1$ digit whence $v_2 = 1$.
- For $n = 2$ we have $s_2 = 2^0 \times 0 + 2^1 \times 1 = 2$ and k sweeping through $\{0, \ldots, 2^2 - 1\} = \{0, 1, 2, 3\}$.
  - For $k = 0$, $v_{s_2+0\times2+1} = v_3$ and $v_{s_2+0\times2+2} = v_4$, so v_3v_4 is the binary representation of $k = 0$ in $n = 2$ digits whence $v_3v_4 = 00$. Thus $v_3 = 0$ and $v_4 = 0$.
  - For $k = 1$, $v_{s_2+1\times2+1} = v_5$ and $v_{s_2+1\times2+2} = v_6$, so v_5v_6 is the binary representation of $k = 1$ in $n = 2$ digits whence $v_5v_6 = 01$. Thus $v_5 = 0$ and $v_6 = 1$.
  - For $k = 2$, $v_{s_2+2\times2+1} = v_7$ and $v_{s_2+2\times2+2} = v_8$, so v_7v_8 is the binary representation of $k = 2$ in $n = 2$ digits whence $v_7v_8 = 10$. Thus $v_7 = 1$ and $v_8 = 0$.
  - For $k = 3$, $v_{s_2+3\times2+1} = v_9$ and $v_{s_2+3\times2+2} = v_{10}$, so v_9v_{10} is the binary representation of $k = 3$ in $n = 2$ digits whence $v_9v_{10} = 11$. Thus $v_9 = 1$ and $v_{10} = 1$.
- For $n = 3$ we have $s_3 = 2^0 \times 0 + 2^1 \times 1 + 2^2 \times 2 = 10$ and k sweeping through $\{0, \ldots, 2^3 - 1\} = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
  - For $k = 0$, $v_{s_3+0\times3+1} = v_{11}$, $v_{s_3+0\times3+2} = v_{12}$ and $v_{s_3+0\times3+3} = v_{13}$, so $v_{11}v_{12}v_{13}$ is the binary representation of $k = 0$ in $n = 3$ digits whence $v_{11}v_{12}v_{13} = 000$. Thus $v_{11} = 0$, $v_{12} = 0$ and $v_{13} = 0$.
  - For $k = 1$, $v_{s_3+1\times3+1} = v_{14}$, $v_{s_3+1\times3+2} = v_{15}$ and $v_{s_3+1\times3+3} = v_{16}$, so $v_{14}v_{15}v_{16}$ is the binary representation of $k = 1$ in $n = 3$ digits whence $v_{14}v_{15}v_{16} = 001$. Thus $v_{14} = 0$, $v_{15} = 0$ and $v_{16} = 1$.
  - For $k = 2$, $v_{s_3+2\times3+1} = v_{17}$, $v_{s_3+2\times3+2} = v_{18}$ and $v_{s_3+2\times3+3} = v_{19}$, so $v_{17}v_{18}v_{19}$ is the binary representation of $k = 2$ in $n = 3$ digits whence $v_{17}v_{18}v_{19} = 010$. Thus $v_{17} = 0$, $v_{18} = 1$ and $v_{19} = 0$.
  - For $k = 3$, $v_{s_3+3\times3+1} = v_{20}$, $v_{s_3+3\times3+2} = v_{21}$ and $v_{s_3+3\times3+3} = v_{22}$, so $v_{20}v_{21}v_{22}$ is the binary representation of $k = 3$ in $n = 3$ digits whence $v_{20}v_{21}v_{22} = 011$. Thus $v_{20} = 0$, $v_{21} = 1$ and $v_{22} = 1$.
  - For $k = 4$, $v_{s_3+4\times3+1} = v_{23}$, $v_{s_3+4\times3+2} = v_{24}$ and $v_{s_3+4\times3+3} = v_{25}$, so $v_{23}v_{24}v_{25}$ is the binary representation of $k = 4$ in $n = 3$ digits whence $v_{23}v_{24}v_{25} = 100$. Thus $v_{23} = 1$, $v_{24} = 0$ and $v_{25} = 0$.
  - For $k = 5$, $v_{s_3+5\times3+1} = v_{26}$, $v_{s_3+5\times3+2} = v_{27}$ and $v_{s_3+5\times3+3} = v_{28}$, so $v_{26}v_{27}v_{28}$ is the binary representation of $k = 5$ in $n = 3$ digits whence $v_{26}v_{27}v_{28} = 101$. Thus $v_{26} = 1$, $v_{27} = 0$ and $v_{28} = 1$.

– For $k=6$, $v_{s_3+6\times 3+1}=v_{29}$, $v_{s_3+6\times 3+2}=v_{30}$ and $v_{s_3+6\times 3+3}=v_{31}$, so $v_{29}v_{30}v_{31}$ is the binary representation of $k=6$ in $n=3$ digits whence $v_{29}v_{30}v_{31}=110$. Thus $v_{29}=1$, $v_{30}=1$ and $v_{31}=0$.

– For $k=7$, $v_{s_3+7\times 3+1}=v_{32}$, $v_{s_3+7\times 3+2}=v_{33}$ and $v_{s_3+7\times 3+3}=v_{34}$, so $v_{32}v_{33}v_{34}$ is the binary representation of $k=7$ in $n=3$ digits whence $v_{32}v_{33}v_{34}=111$. Thus $v_{32}=1$, $v_{33}=1$ and $v_{34}=1$.

Thus the thirty four, initial co-ordinates of v are 0, 1, 0,0, 0,1, 1,0, 1,1, 0,0,0, 0,0,1, 0,1,0, 0,1,1, 1,0,0, 1,0,1, 1,1,0, 1,1,1.

Theorem 3.1 *The element v, as defined above, has dense orbit with respect to B in the Cantor space $\{0,1\}^{\mathbb{N}}$.*

Proof. Let $x=(x_j)_{j\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}}$ be arbitrary. Let U be an arbitrary neighborhood of x in $\{0,1\}^{\mathbb{N}}$. Then there is $n\in\mathbb{N}$ such that $U=\prod_{j\in\mathbb{N}}U_j=U_1\times\cdots\times U_n\times U_{n+1}\times\cdots$, where $U_j\subset\{0,1\}$ for all $j\in\mathbb{N}$, $U_j=\{0,1\}$ for all $j\geq n+1$, and $x_j\in U_j$ for all $j\in\{1,\ldots,n\}$.

Notice that $x_1\cdots x_n$ is the binary representation of k in n digits for some $k\in\{0,1,2,\ldots,2^n-3,2^n-2,2^n-1\}$ whence

$$(x_1,\ldots,x_n)=(v_{s_n+kn+1},\ldots,v_{s_n+kn+n})$$

for some $k\in\{0,\ldots,2^n-1\}$. Hence $B^{s_n+kn}(v)\in\{x_1\}\times\cdots\times\{x_n\}\times\{0,1\}\times\{0,1\}\times\cdots\subset U_1\times\cdots\times U_n\times\{0,1\}\times\{0,1\}\times\cdots=U$. Thus $B^{s_n+kn}(v)\in U\cap$ orb(v,B) whence $U\cap$ orb$(v,B)\neq\emptyset$. Since x was taken arbitrary in $\{0,1\}^{\mathbb{N}}$, orb(v,B) is dense in $\{0,1\}^{\mathbb{N}}$. ■

Floating and trans-floating-point numbers, n, are of the general form $n=-1^{s_m}m2^{-1^{s_e}e}$ where s_m is the sign of the mantissa, m is the absolute value of the mantissa, s_e is the sign of the exponent, and e is the absolute value of the exponent. We can define a hypercyclic trans-floating-point number by defining its elements bijectively with the elements of a hypercyclic binary sequence, as follows. Let v be the hypercyclic binary sequence defined above. Take the even elements of v to be the mantissa and the odd elements to be the exponent, such that: $v_0=s_m$ is the sign of the mantissa and $v_2=m_0,v_4=m_1,v_6=m_2,...$ are successive bits of the mantissa, starting from the least significant bit at m_0. Similarly $v_1=s_e$ is the sign of the exponent and $v_3=e_0,v_5=e_1,v_7=e_2,...$ are the bits of the exponent in order of increasing significance. We identify: $n=0$ with $s_m=0$, all bits of mantissa zero, $s_e=1$, all bits of the exponent unity; $n=\Phi$ with $s_m=1$, all bits of mantissa zero, $s_e=1$, all bits of the exponent unity; $n=\infty$ with $s_m=0$, all bits of mantissa unity, $s_e=0$, all bits of the exponent unity; $n=-\infty$ with $s_m=1$, all bits of mantissa unity, $s_e=1$, all bits of the exponent unity. Thus we obtain a hypercyclic trans-floating-point space that approximates a hypercyclic transreal space. We may map any of the usual data structures used in computer science onto a hypercyclic space by arranging a bijection between a hypercyclic sequence and the bits of the data structure.

4 Philosophy of an Omniscient Observer

We wish to give a logical description of an omniscient observer. Let us start by considering that an observer is a transvector in the logical space $(\mathbb{R}^T)^\mathbb{N}$. Thus every result that is valid concerning possible worlds is also valid when related to the logical space in which points are observers.

An observer, O_k, is a sequence of transreal numbers $\langle o_{k1}, o_{k2}, ..., o_{km} \rangle$. If the value of o_{km} is plus infinity, then O_k is "willing" to consider the atomic proposition, P_m, as absolutely true. If the value of o_{km} is minus infinity, then O_k is "willing" to consider P_m as absolutely false. If the value of o_{km} is a real number, $r > 0$, then O_k is "willing" to consider P_m true in some degree, in such a way that, if $r_1 > r_2$, and r_1 and r_2 are transreal numbers related, respectively, to P_1 and P_2, then P_1 is more true than P_2. If the value of o_{km} is a real number, $r < 0$, then O_k is "willing" to consider P_m false in some degree, in such a way that, if $r_1 < r_2$, and r_1 and r_2 are transreal numbers related, respectively, to P_1 and P_2, then P_1 is more false than P_2. If the value of o_{km} is zero then O_k is "willing" to consider P_m as being false and true simultaneously. If the value of o_{km} is nullity then O_k is "willing" to consider P_m as being neither false nor true. If the value of o_{km} is in the extended-real range $[-\infty, \infty]$ then the observer, O_k, is "willing" to consider P_m to be an "actual proposition," otherwise o_{km} is nullity and P_m is a "pseudo proposition" with no degree of truth or falsehood.

What about the existence of a Universal Observer, an hypercyclic transvector that can access by approximation every other observer? As proved in the present paper, this kind of observer exists in the abstract, and can be seen as a transvector whose coordinates are nullity or rational numbers, or as a transvector whose coordinates are 0 or 1. In both cases, an interesting consideration emerges: A Universal Observer is the mathematical translation of some "consciousness" that has access, by approximation, to all other "consciousnesses." Since an Observer is a sequence of transreal numbers that give us the "degree of belief" that an Observer has in a sequence of "states of affairs" – we can, in a very general way, consider each P_m as a "state of affairs" (an actual proposition or else a pseudo proposition). Then we can stress: the consciousness that has access to all other consciousnesses, via approximation, "operates" with some pseudo propositions, not just with "actual" or classical propositions. In Phenomenological terms: The "state of affairs" of the Universal Observer starts from a configuration of the world that is not objective, but it is, in some sense, an ideal configuration because it contains some pseudo propositions.

A Universal Observer – an Omniscient Observer, since He knows what lies in the Mind of every Observer – is positioned in the Logical Space as some kind of God that can see our thoughts about the world. This observer is, however, not unique. There are infinitely many such observers, positioned infinitely densely in our logical space. However, if we identify all such observers as a single individual then this individual in omnipresent in space, in the sense that He is arbitrarily close to every point in space. Thus omnipresence follows from omniscience in our logical space.

5 Philosophy of Science

We have shown that there are infinitely many universal possible worlds that can be operated on, by mechanical means, to approximate all other possible worlds arbitrarily closely. That is, there are infinitely many *hypothesis engines* that can mechanically generate all possible hypotheses arbitrarily closely. As all actual worlds are, self-evidently, possible, this means there are infinitely many hypothesis engines that can arbitrarily closely describe the world, indeed the universe, we live in. Furthermore, these hypothesis engines are infinitely dense so infinitely many of them lie arbitrarily close to the thoughts of every mathematician, scientist, and philosopher, as well as being arbitrarily close to the operations of every computer. These abstract hypothesis engines can, in principle, access all of our thoughts but how closely can we access them? Can we build experimental engines and couple them with hypothesis engines to conduct science automatically?

6 Conclusion

We explicitly construct two universal worlds that approximate all logically possible worlds in a hypercyclic sequence. A continuous universal world constructs possible worlds with transreal co-ordinates directly. A discrete universal world provides a binary hypercyclic vector which can be used to create transfloating-point co-ordinates that approximate transreal co-ordinates and which, therefore, provides a computable approximation to a universal world. We also discuss the philosophical implications of universal worlds for an omniscient observer and find that omniscience implies omnipresence in our logcial space.

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