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Transquaternions

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Abstract

The quaternions extend the complex numbers and are used in physics and engineering. Division of quaternions by zero is not defined, which limits physical theories and engineering applications. We now introduce transquaternions, which totalise the arithmetical operations of quaternion addition, subtraction, multiplication, and both left and right division. In particular, division of quaternions by zero is allowed.

The transquaternions are homeomorphic to the unit hypersphere or glome, including its interior, together with an isolated point. The 4D interior of the hypersphere is made up of the ordinary quaternions. The 3D surface of the hypersphere is made up of the infinite transquaternions, which are produced by dividing non-zero quaternions by zero. The isolated point, that lies outside the 4D space containing the hypersphere, is the transquaternion nullity, which is produced by dividing zero by zero.

Transquaternions are a separable compact complete metric topological space.

1 Introduction

Transmathematics is a program that seeks to totalise the usual number systems by allowing division by zero. Thus the real numbers are totalised by the transreal numbers [1] [6] and the complex numbers are totalised by the transcomplex numbers [3] [5]. We now introduce the transquaternions as a totalisation of the quaternions. Our method is to replicate the construction given in [3] but to replace complex numbers with quaternions.

That such a substitution is possible, bodes well for a general totalising construction for Cayley-Dickson algebras.

After the technical work of the paper, we discuss possibilities for future work in both mathematics and physics. We conclude with a statement of the most important original results of this paper.

2 Quaternions

Quaternions are a 4-dimensional associative division algebra over the real numbers that contains complex numbers as a subalgebra. The set of quaternions, $\mathbb{H}$, can be written as $\mathbb{H} = \{a + bi + cj + dk;\ a, b, c, d \in \mathbb{R}\}$ where i, j and k are quaternions such that $i^2 = j^2 = k^2 = ijk = -1$.

The multiplication of a quaternion and a real number is commutative, that is for every quaternion x and every real number α it follows that $x\alpha = \alpha x$. Multiplication by real numbers is distributive over quaternion addition, that is for every quaternion x and y and every real number α it follows that $\alpha(x + y) = \alpha x + \alpha y$.

For every non-zero quaternion y there is a quaternion z such that $yz = zy = 1$. Such a number z, called the multiplicative inverse of y, is denoted by y^{-1} or $\frac{1}{y}$. So the multiplications xy^{-1} and $y^{-1}x$ are well defined for every quaternion x and every non-zero quaternion y. Quaternion multiplication is not commutative in general, there are quaternions x and non-zero quaternions y where $xy^{-1} \neq y^{-1}x$. Because of this there are right divisions and left divisions, respectively, $x/y := xy^{-1}$ and $y\backslash x := y^{-1}x$ for every quaternion x and every non-zero quaternion y. Since multiplication between a quaternion and a real number is commutative, when the dividend or the divisor is a real number, the right division and the left division have the same value. Because of this we define $\frac{x}{y} := x/y = y\backslash x$ for every quaternion x and y with $y \neq 0$ where either x or y (or both) are real numbers.

The quaternion conjugate, defined as $\overline{x} := a - bi - cj - dk$ for all $x = a + bi + cj + dk$, satisfies, among others, the properties:

a) $\overline{\overline{x}} = x$,

b) $\overline{x + y} = \overline{x} + \overline{y}$,

c) $\overline{-y} = -\overline{y}$,

d) $\overline{x - y} = \overline{x} - \overline{y}$,

e) $\overline{xy} = \overline{y}\ \overline{x}$,

f) $\overline{y^{-1}} = \overline{y}^{-1}$ if $y \neq 0$ and

g) $\overline{x/y} = \overline{y}\backslash\overline{x}$ and $\overline{y\backslash x} = \overline{x}/\overline{y}$ if $y \neq 0$

for all quaternions x and y.

The quaternion norm, defined as $|x| := \sqrt{a^2 + b^2 + c^2 + d^2}$ for all $x = a + bi + cj + dk$, satisfies, among others, the properties:

a) $|x|$ is real number,

b) $|x| = 0$ if and only if $x = 0$,

c) $|x|^2 = x\overline{x}$,

d) $|\overline{x}| = |x|$,

e) $|xy| = |x||y|$,

f) $|y^{-1}| = |y|^{-1}$ if $y \neq 0$,

g) $|x/y| = |x|/|y| = |y|\backslash|x| = |y\backslash x|$ if $y \neq 0$ and

h) $|x + y| \leq |x| + |y|$

for all quaternions x and y.

For every non-zero quaternion x there are unique $\theta, \phi \in [0, \pi]$ and $\psi \in (-\pi, \pi]$ such that $x = |x|(\cos(\theta) + i\sin(\theta)\cos(\phi) + j\sin(\theta)\sin(\phi)\cos(\psi) + k\sin(\theta)\sin(\phi)\sin(\psi))$.

Notice that $|\cos(\theta) + i\sin(\theta)\cos(\phi) + j\sin(\theta)\sin(\phi)\cos(\psi) + k\sin(\theta)\sin(\phi)\sin(\psi)| = 1$ for all $\theta, \phi \in [0, \pi]$ and $\psi \in (-\pi, \pi]$.

3 Construction of the Transquaternions

Definition 1. Let $T := \{(x, y);\ x \in \mathbb{H},\ y \in \{0, 1\}\}$. Given $(x, y), (w, z) \in T$, we say that $(x, y) \sim (w, z)$, that is, (x, y) is equivalent to (w, z), with respect to $\sim$, if and only if there is a positive $\alpha \in \mathbb{R}$ such that $x = \alpha w$ and $y = \alpha z$.

For each $(x, y) \in T$, let us write the equivalence class of (x, y) as $[x, y]$, that is, $[x, y] := \{(w, z) \in T;\ (w, z) \sim (x, y)\}$. Let us call each element of $T/\sim$, the quotient set of T with respect to $\sim$, a *transquaternion* and let us write this set as $\mathbb{H}^T$, that is, $\mathbb{H}^T := T/\sim$.

Proposition 2. $\mathbb{H}^T$ is well defined. That is, $\sim$ is, in fact, an equivalence relation in T.

Proof. Let $(x, y), (w, z), (u, v) \in T$ such that $(x, y) \sim (w, z)$ and $(w, z) \sim (u, v)$. Then there are positive $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha w$, $y = \alpha z$, $w = \beta u$ and $z = \beta v$.

Firstly since $x = 1 \times x$ and $y = 1 \times y$ it follows that $(x, y) \sim (x, y)$, whence $\sim$ is reflexive.

Secondly since $w = \dfrac{1}{\alpha}x$ and $z = \dfrac{1}{\alpha}y$ it follows that $(w, z) \sim (x, y)$, whence $\sim$ is symmetric.

Thirdly since $x = (\alpha\beta)u$ and $y = (\alpha\beta)v$ it follows that $(x, y) \sim (u, v)$, whence $\sim$ is transitive. $\square$

Proposition 3. $\mathbb{H}^T = \{[u, 1];\ u \in \mathbb{H}\} \cup \{[u, 0];\ u \in \mathbb{H}, |u| = 1\} \cup \{[0, 0]\}$. Furthermore $\{[u, 1];\ u \in \mathbb{H}\}$ and $\{[u, 0];\ u \in \mathbb{H}, |u| = 1\}$ and $\{[0, 0]\}$ are pairwise disjoint sets; and if $x \neq w$ then $[x, 1] \neq [w, 1]$; and if $|x| = 1$ and $|w| = 1$ and $x \neq w$ then $[x, 0] \neq [w, 0]$.

Proof. Let $[x, y] \in \mathbb{H}^T$ be arbitrary. Either $y = 1$ or $y = 0$.

If $y = 1$ then $[x, y] \in \{[u, 1];\ u \in \mathbb{H}\}$.

If $y = 0$ then either $x = 0$ implying $[x, y] = [0, 0]$ or $x \neq 0$ implying $x = |x|\dfrac{x}{|x|}$ and $0 = |x| \times 0$ and $\left|\dfrac{x}{|x|}\right| = 1$, whence $[x, y] = \left[\dfrac{x}{|x|}, 0\right] \in \{[u, 0];\ u \in \mathbb{H}, |u| = 1\}$. $\square$

Definition 4. For all $[x,y],[w,z] \in \mathbb{H}^T$ we define:

a) (addition) If $[x,y],[w,z] \in \{[u,0];\ u \in \mathbb{H}, |u| = 1\}$ then $[x,y] + [w,z] := \left[\frac{x}{|x|} + \frac{w}{|w|}, 0\right]$ otherwise $[x,y] + [w,z] := [xz + wy, yz]$.

b) (multiplication) $[x,y] \times [w,z] := [xw, yz]$.

c) (opposite) $-[x,y] := [-x,y]$.

d) (reciprocal) If $x = 0$ then $[x,y]^{-1} := [y,x]$ otherwise $[x,y]^{-1} := \left[\frac{y}{x}, 1\right]$.

e) (subtraction) $[x,y] - [w,z] := [x,y] + \big(-[w,z]\big)$.

f) (right division) $[x,y]/[w,z] := [x,y] \times [w,z]^{-1}$.

g) (left division) $[w,z]\backslash[x,y] := [w,z]^{-1} \times [x,y]$.

Notice that there is no ambiguity in the division $\frac{y}{x}$ in the item d because y is a real number. The fraction $\frac{y}{x}$ denotes both y/x and $x\backslash y$, since $y/x = x\backslash y$ because y is a real number.

We are conscious that we abuse notation when we reuse the symbols for quaternion arithmetical operations to define the transquaternion arithmetical operations. However, we emphasise that this is not a problem because the context distinguishes the set to which the symbols refer. For example when we say $[x,y]+[w,z] = [wy+xz, yz]$ it is clear that the sign "+" on the left hand side of the equality refers to addition in $\mathbb{H}^T$ while the sign "+" on the right hand side of the equality refers to addition in $\mathbb{H}$. Moreover, as will be seen in Theorem 7 and Remark 8, in a suitable sense, $\mathbb{H}$ is a subset of $\mathbb{H}^T$ and when the operations in $\mathbb{H}^T$ are restricted to $\mathbb{H}$ they coincide with the usual operations of $\mathbb{H}$.

Proposition 5. The operations $+$, $\times$, $-$, $^{-1}$, $/$ and $\backslash$ are well defined. That is, $[x,y] + [w,z]$, $[x,y] \times [w,z]$, $-[w,z]$, $[x,y] - [w,z]$, $[w,z]^{-1}$, $[x,y]/[w,z]$ and $[w,z]\backslash[x,y]$ are independent of the choice of the representatives of the classes $[x,y]$ and $[w,z]$.

Proof. Let $[x,y],[w,z] \in \mathbb{H}^T$, $(x',y') \in [x,y]$ and $(w',z') \in [w,z]$. We have that there are positive $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha x'$, $y = \alpha y'$, $w = \beta w'$ and $z = \beta z'$.

If $[x,y],[w,z] \in \{[u,0];\ u \in \mathbb{H}, |u| = 1\}$, then $x \neq 0$, $w \neq 0$, $y = 0$ and $z = 0$, whence $x' \neq 0$, $w' \neq 0$, $y' = 0$, $z' = 0$, $|x| = \alpha|x'|$ and $|w| = \beta|w'|$. Thus $\frac{x}{|x|} = \frac{\alpha x'}{\alpha|x'|} = \frac{x'}{|x'|}$ and $\frac{w}{|w|} = \frac{\beta w'}{\beta|w'|} = \frac{w'}{|w'|}$, whence $\left(\frac{x}{|x|} + \frac{w}{|w|}, 0\right) = \left(\frac{x'}{|x'|} + \frac{w'}{|w'|}, 0\right)$. Otherwise $xz + wy = \alpha x'\beta z' + \beta w'\alpha y' = (\alpha\beta)(x'z' + w'y')$ and $yz = \alpha y'\beta z' = (\alpha\beta)(y'z')$, whence $(xz + wy, yz) \sim (x'z' + w'y', y'z')$. Hence, in both cases, $[x,y] + [w,z] = [x',y'] + [w',z']$. Therefore addition is well defined.

Since $xw = \alpha x'\beta w' = (\alpha\beta)(x'w')$ and $yz = \alpha y'\beta z' = (\alpha\beta)(y'z')$ it follows that $(xw, yz) \sim (x'w', y'z')$, whence $[x,y]\times[w,z] = [x',y']\times[w',z']$. Therefore multiplication is well defined.

Since $-w = -(\beta w') = \beta(-w')$ and $z = \beta z'$ it follows that $(-w, z) \sim (-w', z')$, whence $-[w, z] = -[w', z']$. Therefore the opposite is well defined.

If $w = 0$ then $w' = 0$ and $w = 0 = \beta \times 0 = \beta w'$ and $z = \beta z'$, whence $(z, w) \sim (z', w')$ and, thereby, $[w, z]^{-1} = [w', z']^{-1}$. If $w \neq 0$ then $\dfrac{z}{w} = \dfrac{\beta z'}{\beta w'} = \dfrac{z'}{w'}$, whence $\left(\dfrac{z}{w}, 1\right) = \left(\dfrac{z'}{w'}, 1\right)$. Therefore the reciprocal is well defined.

Notice that subtraction, right division and left division are well defined by consequence of the four previous operations. $\square$

Theorem 6. There are no exceptions in any of the transquaternion arithmetical operations: addition, opposite, subtraction, multiplication, reciprocal, right division and left division. That is, for all $[x, y], [w, z] \in \mathbb{H}^T$ it follows that $[x, y] + [w, z]$, $-[w, z]$, $[x, y] - [w, z]$, $[x, y] \times [w, z]$, $[w, z]^{-1}$, $[x, y]/[w, z]$ and $[w, z]\backslash[x, y]$ are well defined elements of $\mathbb{H}^T$.

Proof. The result is immediate from the definitions of the transquaternion arithmetical operations (Definition 4). $\square$

Theorem 7. The set $H := \big\{[u, 1];\ u \in \mathbb{H}\big\}$ is a 4-dimensional associative division algebra over the real numbers.

Proof. Firstly of all notice that $[x, 1]+[y, 1] = [x+y, 1]$ and $[x, 1]\times[y, 1] = [xy, 1]$ for any $x, y \in \mathbb{H}$.

Denote $R := \big\{[\alpha, 1];\ \alpha \in \mathbb{R}\big\}$ and define for all $[\alpha, 1], [\beta, 1] \in R$: $[\alpha, 1] \leq [\beta, 1]$ if and only if $\alpha \leq \beta$. Since the function $\overline{\pi} : \mathbb{R} \to R$, where $\overline{\pi}(\alpha) = [\alpha, 1]$ for all $\alpha \in \mathbb{R}$, is bijective and

- $\overline{\pi}(\alpha) + \overline{\pi}(\beta) = \overline{\pi}(\alpha + \beta)$,
- $\overline{\pi}(\alpha) \times \overline{\pi}(\beta) = \overline{\pi}(\alpha\beta)$ and
- $\overline{\pi}(\alpha) \leq \overline{\pi}(\beta)$ if and only if $\alpha \leq \beta$

for all $\alpha, \beta \in \mathbb{R}$ and $\mathbb{R}$ is a complete ordered field it follows that R is a complete ordered field and $\overline{\pi}$ is an isomorphism of ordered fields.

Since the function $\pi : \mathbb{H} \to H$, where $\pi(x) = [x, 1]$ for all $x \in \mathbb{H}$, is bijective and

- $\pi(x) + \pi(w) = \pi(x + w)$ and
- $\pi(x) \times \pi(w) = \pi(xw)$

for all $x, w \in \mathbb{H}$ and $\mathbb{H}$ is a 4-dimensional associative division algebra over $\mathbb{R}$ it follows that H is a 4-dimensional associative division algebra over the field R. Therefore, since R is $\mathbb{R}$ up to isomorphism, H is a 4-dimensional associative division algebra over the real numbers. $\square$

Remark 8. Notice that since π is an isomorphism of associative division algebras between H and $\mathbb{H}$, we may say that H is a "copy" of $\mathbb{H}$ in $\mathbb{H}^T$. Therefore we may allow an abuse of language and notation: each $[x, 1] \in H$ will be written, merely, as x, and H will be called the set of quaternions. In this sense we may say that

$$\mathbb{H} \subset \mathbb{H}^T.$$

Furthermore there is now no ambiguity in the use of the arithmetical symbols $+$, $\times$, $-$, $^{-1}$, $/$ and $\backslash$ so we can consider that all arithmetical symbols refer to transquaternion arithmetical operations, since the isomorphism π ensures that when they are applied to quaternions, their results coincide with the results of their homologous quaternion arithmetical operations.

4 Transquaternions as Fractions

We already know that $x/y = y\backslash x$ for every quaternion x and y with $y \neq 0$ where either $x \in \mathbb{R}$ or $y \in \mathbb{R}$. Theorem 9 says that the above sentence is still true if we replace the word "quaternion" by the word "transquaternion". Furthermore, from the Theorem 6 there is no exception in the right division and in the left division. In particular every transquaternion can be divided by zero. So, Theorem 9 also allows us to remove the restriction about y in the above sentence.

Theorem 9. Let $a, b \in \mathbb{H}^T$. If $a \in \mathbb{R}$ or $b \in \mathbb{R}$ then $a/b = b\backslash a$.

Proof. Let $a, b \in \mathbb{H}^T$.

If $a \in \mathbb{R}$ then $a = [a, 1]$. There are $w \in \mathbb{H}$ and $z \in \{0, 1\}$ such that $b = [w, z]$. If $w \neq 0$ then $\frac{z}{w} \in \mathbb{H}$ whence, since $a \in \mathbb{R}$, $a\frac{z}{w} = \frac{z}{w}a$ and, therefore, $a/b = [a, 1]/[w, z] = [a, 1] \times [w, z]^{-1} = [a, 1] \times \left[\frac{z}{w}, 1\right] = \left[a\frac{z}{w}, 1 \times 1\right] = \left[\frac{z}{w}a, 1 \times 1\right] = \left[\frac{z}{w}, 1\right] \times [a, 1] = [w, z]^{-1} \times [a, 1] = [w, z]\backslash[a, 1] = b\backslash a$. If $w = 0$ then $a/b = [a, 1]/[w, z] = [a, 1] \times [w, z]^{-1} = [a, 1] \times [0, z]^{-1} = [a, 1] \times [z, 0] = [a \times z, 1 \times 0] = [z \times a, 0 \times 1] = [z, 0] \times [a, 1] = [0, z]^{-1} \times [a, 1] = [w, z]^{-1} \times [a, 1] = [w, z]\backslash[a, 1] = b\backslash a$.

If $b \in \mathbb{R}$ then $b = [b, 1]$. There are $x \in \mathbb{H}$ and $y \in \{0, 1\}$ such that $a = [x, y]$. If $b \neq 0$ then $\frac{1}{b} \in \mathbb{R}$ whence $x\frac{1}{b} = \frac{1}{b}x$ and, therefore, $a/b = [x, y]/[b, 1] = [x, y] \times [b, 1]^{-1} = [x, y] \times \left[\frac{1}{b}, 1\right] = \left[x\frac{1}{b}, y \times 1\right] = \left[\frac{1}{b}x, 1 \times y\right] = \left[\frac{1}{b}, 1\right] \times [x, y] = [b, 1]^{-1} \times [x, y] = [b, 1]\backslash[x, y] = b\backslash a$. If $b = 0$ then $a/b = a/0 = [x, y]/[0, 1] = [x, y] \times [0, 1]^{-1} = [x, y] \times [1, 0] = [x \times 1, y \times 0] = [1 \times x, 0 \times y] = [1, 0] \times [x, y] = [0, 1]^{-1} \times [x, y] = [0, 1]\backslash[x, y] = 0\backslash a = b\backslash a$. $\square$

Remark 10. Theorem 9 allows us to define $\frac{a}{b} := a/b = b\backslash a$ for every transquaternions a and b where either a or b (or both) is a real number.

Theorem 11. Every transquaternion can be written as a fraction between quaternions. More specifically, for all $a \in \mathbb{H}^T$ there are $u \in \mathbb{H}$ and $v \in \{0, 1\}$ such that $a = \frac{u}{v}$.

Notice that the division in the fraction $\frac{u}{v}$ is the transquaternion division (Definition 4). That is, u and v are **quaternions** but the division in the fraction $\frac{u}{v}$ is the **transquaternion** division. This is not a problem since all quaternions are transquaternions so that transquaternion division applies to quaternions.

Proof. Let $a \in \mathbb{H}^T$. There are $u \in \mathbb{H}$ and $v \in \{0,1\}$ such that $a = [u,v]$. If $v = 0$ then $[v,1]^{-1} = [1,v]$ otherwise $[v,1]^{-1} = \left[\frac{1}{v}, 1\right]$ and $v = 1$ whence $[v,1]^{-1} = \left[\frac{1}{v}, 1\right] = \left[\frac{1}{1}, 1\right] = [1,1] = [1,v]$. That is, in any case, $[v,1]^{-1} = [1,v]$. Thus $\frac{u}{v} = u/v = [u,1]/[v,1] = [u,1] \times [v,1]^{-1} = [u,1] \times [1,v] = [u \times 1, 1 \times v] = [u,v] = a$. $\square$

Remark 12. Because of Theorem 11, the transquaternion equivalence (Definition 1) and arithmetical operations (Definition 4) can be performed as operations on fractions. Let $x, w \in \mathbb{H}$ and $y, z \in \{0,1\}$.

a) $\frac{x}{y} = \frac{w}{z}$ if and only if there is a positive $\alpha \in \mathbb{R}$ such that $x = \alpha w$ and $y = \alpha z$.

b) If $x \neq 0$, $w \neq 0$, $y = 0$ and $z = 0$ then $\frac{x}{y} + \frac{w}{z} = \frac{\frac{x}{|x|} + \frac{w}{|w|}}{0}$.

Otherwise $\frac{x}{y} + \frac{w}{z} = \frac{xz + wy}{yz}$.

c) $-\frac{w}{z} = \frac{-w}{z}$.

d) $\frac{x}{y} - \frac{w}{z} = \frac{x}{y} + \frac{-w}{z}$.

e) $\frac{x}{y} \times \frac{w}{z} = \frac{xw}{yz}$.

f) If $w = 0$ then $\left(\frac{w}{z}\right)^{-1} = \frac{z}{w}$.

Otherwise, $\left(\frac{w}{z}\right)^{-1} = \frac{\frac{z}{w}}{1}$.

g) $\left(\frac{x}{y}\right) / \left(\frac{w}{z}\right) = \frac{x}{y} \times \left(\frac{w}{z}\right)^{-1}$.

h) $\left(\frac{w}{z}\right) \backslash \left(\frac{x}{y}\right) = \left(\frac{w}{z}\right)^{-1} \times \frac{x}{y}$.

For practical purposes it is not necessary to know that transquaternions are equivalence classes of ordered pairs of quaternions. Section 3 is an explicit and rigorous construction of the new numbers, transquaternions, from the already known numbers, quaternions, in order to prove the consistency of the arithmetic of transquaternions. However, for practical purposes, it is enough to know that transquaternions are fractions of quaternions (Theorem 11) and that the arithmetic of fractions, expressed in (Remark 12), holds.

Theorem 13. It follows that

$$\mathbb{H}^T = \{x/y;\ x, y \in \mathbb{H}\} = \{x \backslash y;\ x, y \in \mathbb{H}\}.$$

Proof. By Theorem 11, $\mathbb{H}^T \subset \{x/y;\ x, y \in \mathbb{H}\}$. Let x and y be arbitrary quaternions. Since $\mathbb{H} \subset \mathbb{H}^T$ it follows that $x, y \in \mathbb{H}^T$, whence, by Theorem 6, $x/y \in \mathbb{H}^T$. As x and y were taken arbitrarily in $\mathbb{H}$ it follows that $\{x/y;\ x, y \in \mathbb{H}\} \subset \mathbb{H}^T$. Therefore $\mathbb{H}^T = \{x/y;\ x, y \in \mathbb{H}\}$.

Similarly, we have that $\mathbb{H}^T = \{x \backslash y;\ x, y \in \mathbb{H}\}$. $\square$

5 Notable Subsets of Transquaternions

Theorem 14. The transcomplex numbers are a subset of the transquaternions,

$$\mathbb{C}^T \subset \mathbb{H}^T,$$

and every transquaternion arithmetical operation coincides with its homologous transcomplex one when applied to transcomplex numbers, where that right and left divisions coincide with each other in transcomplex numbers.

Proof. In [5] we see that $\mathbb{C}^T = \left\{\frac{x}{y};\ x, y \in \mathbb{C}\right\}$. Since $\mathbb{C} \subset \mathbb{H}$ it follows that $\left\{\frac{x}{y};\ x, y \in \mathbb{C}\right\} \subset \{x/y;\ x, y \in \mathbb{H}\}$. Therefore $\mathbb{C}^T = \left\{\frac{x}{y};\ x, y \in \mathbb{C}\right\} \subset \{x/y;\ x, y \in \mathbb{H}\} = \mathbb{H}^T$.

Since the definitions of the transquaternion arithmetical operations are identical to the definitions of the corresponding transcomplex operations ([3], Definition 2), where the right and left divisions coincide with each other in transcomplex numbers, it follows that the transquaternion arithmetical operations coincide with their homologous transcomplex ones when applied to transcomplex numbers. □

Theorem 15. The transreal numbers are a subset of the transquaternions, $\mathbb{R}^T \subset \mathbb{H}^T$, and every transquaternion arithmetical operation coincide with its homologous transreal one when applied to transreal numbers.

Proof. In [3] we see that $\mathbb{R}^T \subset \mathbb{C}^T$. Thus $\mathbb{R}^T \subset \mathbb{C}^T \subset \mathbb{H}^T$, whence $\mathbb{R}^T \subset \mathbb{H}^T$.

Since every transquaternion arithmetical operation coincides with its homologous transcomplex one when applied to transcomplex numbers and all transreal numbers are transcomplex numbers, it follows that every transquaternion arithmetical operation coincides with its homologous transcomplex one when applied to transreal numbers. But the transcomplex arithmetical operations are the transreal ones when applied to transreal numbers. Because of this every transquaternion arithmetical operation coincide with its homologous transreal one when applied to transreal numbers. □

The rectangle ABCD represents the set of real numbers, $\mathbb{R}$.
The rectangle GBCH represents the set of hyperreal numbers, ${}^*\mathbb{R}$.
The rectangle AIJD represents the set of complex numbers, $\mathbb{C}$.
The rectangle EFCD represents the set of transreal numbers, $\mathbb{R}^T$.
The rectangle AILK represents the set of quaternions, $\mathbb{H}$.
The rectangle EMJD represents the set of transcomplex numbers, $\mathbb{C}^T$.
The rectangle ENOK represents the set of transquaternions, $\mathbb{H}^T$.

Despite the fact that the hyperreal numbers are not part of the subject of the present paper, we mention them in Figure 1 to emphasise that the strictly hyperreal numbers have nothing in common with the strict

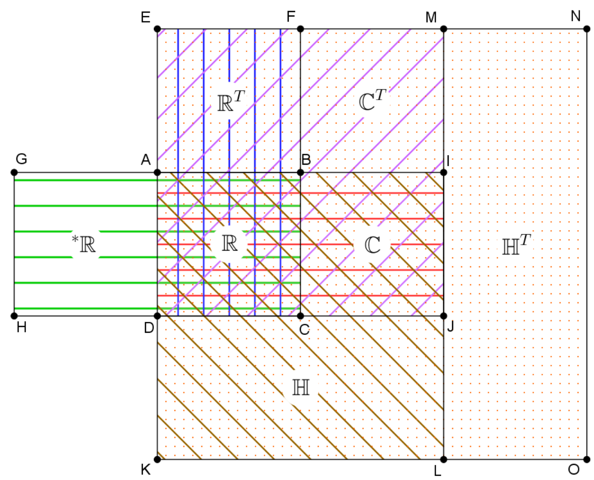

Figure 1: Notable subsets of the transquaternions, together with the hyperreals.

transquaternions. In particular, none of the hyperreal infinite numbers is an infinite transquaternion and none of the infinite transquaternions is a hyperreal infinite number.

Definition 16. Now let us define *infinity* and *nullity*, respectively, by $\infty := [1, 0]$ and $\Phi := [0, 0]$.

Theorem 17. Transquaternion infinity coincides with transcomplex infinity and transquaternion nullity coincides with transcomplex nullity.

Proof. The results follows from the fact that the definition of transquaternion infinity and transquaternion nullity are, respectively, identical to the definition of transcomplex infinity and transcomplex nullity [3]. □

Remark 18. Because of Theorem 17 there is no ambiguity in writing the number ∞ or the number Φ. It does not matter whether ∞ denotes the transreal infinity or transcomplex infinity or transquaternion infinity since these three numbers are the same. Likewise it does not matter whether Φ denotes the transreal nullity or transcomplex nullity or transquaternion nullity since these three numbers are the same.

Remark 19. $[0, \infty]$ denotes the interval of transreal numbers $\{x \in \mathbb{R}^T;\ 0 \leq x \leq \infty\}$ [6].

Definition 20. Henceforth, for each $\theta, \phi \in [0, \pi]$ and $\psi \in (-\pi, \pi]$, we denote $\mathcal{A}(\theta, \phi, \psi) := \cos(\theta) + i\sin(\theta)\cos(\phi) + j\sin(\theta)\sin(\phi)\cos(\psi) + k\sin(\theta)\sin(\phi)\sin(\psi)$.

Theorem 21. It follows that

$$\mathbb{H}^T = \{r\mathcal{A}(\theta, \phi, \psi);\quad r \in [0, \infty] \cup \{\Phi\},\ \theta, \phi \in [0, \pi],\ \psi \in (-\pi, \pi]\}$$

and

a) $0\mathcal{A}(\theta_1,\phi_1,\psi_1) = 0\mathcal{A}(\theta_2,\phi_2,\psi_2)$ for all $\theta_1,\theta_2,\phi_1,\phi_2 \in [0,\pi]$ and $\psi_1,\psi_2 \in (-\pi,\pi]$,

b) $\Phi\mathcal{A}(\theta_1,\phi_1,\psi_1) = \Phi\mathcal{A}(\theta_2,\phi_2,\psi_2)$ for all $\theta_1,\theta_2,\phi_1,\phi_2 \in [0,\pi]$ and $\psi_1,\psi_2 \in (-\pi,\pi]$,

c) $\infty\mathcal{A}(\theta_1,\phi_1,\psi_1) = \infty\mathcal{A}(\theta_2,\phi_2,\psi_2)$ if and only if $\theta_1=\theta_2$, $\phi_1=\phi_2$ and $\psi_1=\psi_2$ and

d) $r\mathcal{A}(\theta_1,\phi_1,\psi_1) = r\mathcal{A}(\theta_2,\phi_2,\psi_2)$ if and only if $\theta_1=\theta_2$, $\phi_1=\phi_2$ and $\psi_1=\psi_2$, when $r\in(0,\infty)$.

Proof. Let $z\in\mathbb{H}^T$. By Proposition 3, $\mathbb{H}^T = \big\{[u,1];\ u\in\mathbb{H}\big\}\cup\big\{[u,0];\ u\in\mathbb{H}, |u|=1\big\}\cup\big\{[0,0]\big\}$. Thus either $z\in\big\{[u,1];\ u\in\mathbb{H}\big\}$ or $z\in\big\{[u,0];\ u\in\mathbb{H}, |u|=1\big\}$ or $z=[0,0]$.

- If $z\in\big\{[u,1];\ u\in\mathbb{H}\big\}$ then $z\in\mathbb{H}$. If $z=0$ then for all $\theta,\phi\in[0,\pi]$ and $\psi\in(-\pi,\pi]$, $\mathcal{A}(\theta,\phi,\psi)$ is a quaternion, whence $0\mathcal{A}(\theta,\phi,\psi)=0$. Thus $z=0=0\mathcal{A}(\theta,\phi,\psi)$. If $z\neq 0$ then there are $r\in(0,\infty)$ and $\theta,\phi\in[0,\pi]$ and $\psi\in(-\pi,\pi]$ such that $z=r\mathcal{A}(\theta,\phi,\psi)$.
- If $z\in\big\{[u,0];\ u\in\mathbb{H}, |u|=1\big\}$ then there is $u\in\mathbb{H}$ where $|u|=1$ such that $z=[u,0]=[1\times u, 0\times 1]=[1,0]\times[u,1]$. Since $|u|=1$, $u\neq 0$, whence there are $\theta,\phi\in[0,\pi]$ and $\psi\in(-\pi,\pi]$ such that $u=|u|\mathcal{A}(\theta,\phi,\psi)$. Thus $z=[1,0]\times[u,1]=\infty\times u=\infty\times(|u|\mathcal{A}(\theta,\phi,\psi))=\infty\times(1\times\mathcal{A}(\theta,\phi,\psi))=\infty\mathcal{A}(\theta,\phi,\psi)$.
- If $z=[0,0]$ then for all $\theta,\phi\in[0,\pi]$ and $\psi\in(-\pi,\pi]$, $\mathcal{A}(\theta,\phi,\psi)$ is a quaternion, whence $\mathcal{A}(\theta,\phi,\psi)=[\mathcal{A}(\theta,\phi,\psi),1]$. Thus $z=[0,0]=[0\times\mathcal{A}(\theta,\phi,\psi),0\times 1]=[0,0]\times[\mathcal{A}(\theta,\phi,\psi),1]=\Phi\mathcal{A}(\theta,\phi,\psi)$.

Thus $\mathbb{H}^T\subset\{r\mathcal{A}(\theta,\phi,\psi);\ \ r\in[0,\infty]\cup\{\Phi\},\ \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\}$.

If $r\in[0,\infty]\cup\{\Phi\}$ and $\theta,\phi\in[0,\pi]$ and $\psi\in(-\pi,\pi]$ then $r\in[0,\infty]\cup\{\Phi\}\subset\mathbb{R}^T\subset\mathbb{H}^T$ and $\mathcal{A}(\theta,\phi,\psi)\in\mathbb{H}\subset\mathbb{H}^T$, whence $r\mathcal{A}(\theta,\phi,\psi)\in\mathbb{H}^T$. Thus $\{r\mathcal{A}(\theta,\phi,\psi);\ \ r\in[0,\infty]\cup\{\Phi\},\ \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\}\subset\mathbb{H}^T$.

a) Proved above.

b) Proved above.

c) If $\theta_1=\theta_2$, $\phi_1=\phi_2$ and $\psi_1=\psi_2$ then clearly $\infty\mathcal{A}(\theta_1,\phi_1,\psi_1)=\infty\mathcal{A}(\theta_2,\phi_2,\psi_2)$. If $\infty\mathcal{A}(\theta_1,\phi_1,\psi_1)=\infty\mathcal{A}(\theta_2,\phi_2,\psi_2)$ then $[\mathcal{A}(\theta_1,\phi_1,\psi_1),0]=[1\times\mathcal{A}(\theta_1,\phi_1,\psi_1),0\times 1]=[1,0]\times[\mathcal{A}(\theta_1,\phi_1,\psi_1),1]$ $=\infty\mathcal{A}(\theta_1,\phi_1,\psi_1)=\infty\mathcal{A}(\theta_2,\phi_2,\psi_2)=[1,0]\times[\mathcal{A}(\theta_2,\phi_2,\psi_2),1]=[1\times\mathcal{A}(\theta_2,\phi_2,\psi_2),0\times 1]=[\mathcal{A}(\theta_2,\phi_2,\psi_2),0]$, whence there is a positive $\alpha\in\mathbb{R}$ such that $\mathcal{A}(\theta_1,\phi_1,\psi_1)=\alpha\mathcal{A}(\theta_2,\phi_2,\psi_2)$. Hence $1=|\mathcal{A}(\theta_1,\phi_1,\psi_1)|=|\alpha\mathcal{A}(\theta_2,\phi_2,\psi_2)|=|\alpha||\mathcal{A}(\theta_2,\phi_2,\psi_2)|=|\alpha|\times 1=|\alpha|=\alpha$. Thus $\mathcal{A}(\theta_1,\phi_1,\psi_1)=\alpha\mathcal{A}(\theta_2,\phi_2,\psi_2)=1\times\mathcal{A}(\theta_2,\phi_2,\psi_2)=\mathcal{A}(\theta_2,\phi_2,\psi_2)$, whence $\theta_1=\theta_2$, $\phi_1=\phi_2$ and $\psi_1=\psi_2$.

d) The result follows from the uniqueness of the polar representation of the non-zero quaternions.

□

Notice that all transquaternions have a unique polar description, except zero and nullity.

Definition 22. We denote

$$\mathbb{H}^T_\infty := \left\{\infty\mathcal{A}(\theta,\phi,\psi);\ \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\right\}.$$

Theorem 23. It follows that

$$\mathbb{H}^T = \mathbb{H}\cup\mathbb{H}^T_\infty\cup\{\Phi\}.$$

Proof. By Theorem 21,

$$\mathbb{H}^T = \{r\mathcal{A}(\theta,\phi,\psi);\quad r\in[0,\infty]\cup\{\Phi\},\ \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\}\,,$$

whence

$$\begin{array}{rcl}\mathbb{H}^T & = & \{r\mathcal{A}(\theta,\phi,\psi);\quad r\in[0,\infty]\cup\{\Phi\},\ \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\}\\ & = & \{r\mathcal{A}(\theta,\phi,\psi);\quad r\in[0,\infty),\ \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\}\\ & \cup & \{\infty\mathcal{A}(\theta,\phi,\psi);\quad \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\}\\ & \cup & \{\Phi\mathcal{A}(\theta,\phi,\psi);\quad \theta,\phi\in[0,\pi],\ \psi\in(-\pi,\pi]\}\\ & = & \mathbb{H}\cup\mathbb{H}^T_\infty\cup\{\Phi\}.\end{array}$$

□

Definition 24. Let us refer to the elements of $\mathbb{H}$ as *finite transquaternions*, to the elements of $\mathbb{H}^T_\infty\cup\{\Phi\}$ as *non-finite transquaternions* or *strict transquaternions* and, particularly, to the elements of $\mathbb{H}^T_\infty$ as *infinite transquaternions*.

6 Properties of Transquaternions

Theorem 25. Elementary properties of transquaternion arithmetic.

a) The sum of nullity with any transquaternion is nullity: $\Phi + z = z + \Phi = \Phi$ for all $z\in\mathbb{H}^T$.

b) The sum of any non-opposite, infinite transquaternions is an infinite transquaternion: If $z,w\in\mathbb{H}^T_\infty$ and $z\neq -w$ then $z+w\in\mathbb{H}^T_\infty$.

c) The sum of opposite, infinite transquaternions is nullity: If $z,w\in\mathbb{H}^T_\infty$ and $z=-w$ then $z+w=\Phi$.

d) The sum of an infinite transquaternion with a finite transquaternion is the infinite transquaternion: If $z\in\mathbb{H}^T_\infty$ and $w\in\mathbb{H}$ then $z+w=w+z=z$.

e) The opposite of nullity is nullity: $-\Phi=\Phi$.

f) Subtraction of a non-finite transquaternion from itself is nullity: If $z\in\mathbb{H}^T\setminus\mathbb{H}$ then $z-z=\Phi$.

g) The product of nullity with any transquaternion is nullity: $\Phi\times z = z\times\Phi=\Phi$ for all $z\in\mathbb{H}^T$.

h) The product of any infinite transquaternions is an infinite transquaternion: If $z,w\in\mathbb{H}^T_\infty$ then $z\times w\in\mathbb{H}^T_\infty$.

i) The product of an infinite transquaternion with a non-zero, finite transquaternion is an infinite transquaternion: If $z\in\mathbb{H}^T_\infty$ and $w\in\mathbb{H}\setminus\{0\}$ then $z\times w, w\times z\in\mathbb{H}^T_\infty$.

j) The product of an infinite transquaternion with zero is nullity: If $z \in \mathbb{H}^T_\infty$ then $z \times 0 = 0 \times z = \Phi$.

k) The reciprocal of nullity is nullity: $\Phi^{-1} = \Phi$.

l) The reciprocal of zero is infinity: $0^{-1} = \infty$.

m) The reciprocal of any infinite transquaternion is zero: If $z \in \mathbb{H}^T_\infty$ then $z^{-1} = 0$.

n) Division of non-finite transquaternions is nullity: If $z, w \cup \mathbb{H}^T \setminus \mathbb{H}$ then $z/w = w\backslash z = \Phi$.

o) Zero divided by zero is nullity: $\dfrac{0}{0} = \Phi$.

Proof. Let $z, w \in \mathbb{H}^T$ be arbitrary. There are $x, u \in \mathbb{H}$ and $y, t \in \{0, 1\}$ such that $z = \dfrac{x}{y}$ and $w = \dfrac{u}{t}$.

a) $\Phi + z = \dfrac{0}{0} + \dfrac{x}{y} = \dfrac{0 \times y + x \times 0}{0 \times y} = \dfrac{0}{0} = \Phi$. Similarly $z + \Phi = \Phi$.

b) If $z, w \in \mathbb{H}^T_\infty$ and $z \neq -w$ then $x \neq 0$, $u \neq 0$, $y = 0$, $t = 0$ and $\dfrac{x}{|x|} + \dfrac{u}{|u|} \neq 0$, whence $z + w = \dfrac{x}{0} + \dfrac{u}{0} = \dfrac{\frac{x}{|x|} + \frac{u}{|u|}}{0} \in \mathbb{H}^T_\infty$.

c) If $z, w \in \mathbb{H}^T_\infty$ and $z = -w$ then $x \neq 0$, $u \neq 0$, $y = 0$, $t = 0$ and $\dfrac{x}{|x|} + \dfrac{u}{|u|} \neq 0$, whence $z + w = \dfrac{x}{0} + \dfrac{u}{0} = \dfrac{\frac{x}{|x|} + \frac{u}{|u|}}{0} = \dfrac{0}{0} = \Phi$.

d) If $z \in \mathbb{H}^T_\infty$ and $w \in \mathbb{H}$ then $x \neq 0$, $y = 0$, $t \neq 0$, whence $z + w = \dfrac{x}{0} + \dfrac{u}{1} = \dfrac{x \times 1 + u \times 0}{0 \times 1} = \dfrac{x}{0} = z$. Similarly $w + z = z$.

e) $-\Phi = -\dfrac{0}{0} = \dfrac{-0}{0} = \dfrac{0}{0} = \Phi$.

f) The result follows from items (a), (c) and (e).

g) $\Phi \times z = \dfrac{0}{0} \times \dfrac{x}{y} = \dfrac{0 \times x}{0 \times y} = \dfrac{0}{0} = \Phi$. Similarly $z \times \Phi = \Phi$.

h) If $z, w \in \mathbb{H}^T_\infty$ then $x \neq 0$, $u \neq 0$, $y = 0$, $t = 0$ and $xu \neq 0$, whence $z \times w = \dfrac{x}{0} \times \dfrac{u}{0} = \dfrac{xu}{0 \times 0} = \dfrac{xu}{0} \in \mathbb{H}^T_\infty$.

i) If $z \in \mathbb{H}^T_\infty$ and $w \in \mathbb{H} \setminus \{0\}$ then $x \neq 0$, $y = 0$, $u \neq 0$, whence $z \times w = \dfrac{x}{0} \times \dfrac{u}{t} = \dfrac{x \times u}{0 \times t} = \dfrac{xu}{0} \in \mathbb{H}^T_\infty$. Similarly $w \times z \in \mathbb{H}^T_\infty$.

j) If $z \in \mathbb{H}^T_\infty$ then $y = 0$, whence $z \times 0 = \dfrac{x}{0} \times \dfrac{0}{1} = \dfrac{x \times 0}{0 \times 1} = \dfrac{0}{0} = \Phi$. Similarly $0 \times z = \Phi$.

k) $\Phi^{-1} = \left(\dfrac{0}{0}\right)^{-1} = \dfrac{0}{0} = \Phi$.

l) $0^{-1} = \left(\dfrac{0}{1}\right)^{-1} = \dfrac{1}{0} = \infty$.

m) If $z \in \mathbb{H}^T_\infty$ then $z^{-1} = \left(\dfrac{x}{0}\right)^{-1} = \dfrac{\frac{0}{x}}{1} = \dfrac{0}{1} = 0$.

n) The result follows from items (g), (j), (k) and (l).

o) The result follows from items (j) and (l).

$\square$

Definition 26. Given $z \in \mathbb{H}^T$ take $x \in \mathbb{H}$ and $y \in \{0, 1\}$ such that $z = \dfrac{x}{y}$ and define $\overline{z} := \dfrac{\overline{x}}{y}$. We call $\overline{z}$ the *conjugate* of the transquaternion z.

We are again abusing notation when we reuse the symbol for the conjugate of quaternions to define conjugate in $\mathbb{H}^T$. However, this is not a problem because the context distinguishes the set to which the symbols refer. When we say that $\overline{z} = \dfrac{\overline{x}}{y}$ it is clear that the symbol "$^-$" on the left hand side of the equality refers to conjugate in $\mathbb{H}^T$ while the symbols "$^-$" on the right hand side of the equality refer to conjugate in $\mathbb{H}$. Moreover, when the operation of taking the conjugate in $\mathbb{H}^T$ is restricted to $\mathbb{H}$ it coincides with the usual conjugate in $\mathbb{H}$.

Theorem 27. The conjugate of a transquaternion is well defined. That is, the conjugate is independent of the choice of the fraction which represents the transquaternion. In other words, if $x, w \in \mathbb{H}$ and $y, t \in \{0, 1\}$ and $\dfrac{x}{y} = \dfrac{w}{t}$ then $\overline{\left(\dfrac{x}{y}\right)} = \overline{\left(\dfrac{w}{t}\right)}$.

Proof. Let $x, w \in \mathbb{H}$ and $y, t \in \{0, 1\}$ such that $\dfrac{x}{y} = \dfrac{w}{t}$. If $y = 1$ then $t = 1$, whence $x = w$ and the result is immediate. If $y = 0$ and $x = 0$ then $t = 0$ and $w = 0$, whence the result is also immediate. If $y = 0$ and $x \neq 0$ then $t = 0$ and $w \neq 0$ and $\dfrac{x}{|x|} = \dfrac{w}{|w|} \in \mathbb{H}$, whence $\dfrac{\overline{x}}{|\overline{x}|} = \dfrac{\overline{w}}{|\overline{w}|}$. Thus, $\overline{\left(\dfrac{x}{y}\right)} = \overline{\left(\dfrac{x}{0}\right)} = \dfrac{\overline{x}}{0} = \dfrac{\frac{\overline{x}}{|\overline{x}|}}{0} = \dfrac{\frac{\overline{w}}{|\overline{w}|}}{0} = \dfrac{\overline{w}}{0} = \overline{\left(\dfrac{w}{0}\right)} = \overline{\left(\dfrac{w}{t}\right)}$. $\square$

Of course, when $z \in \mathbb{H}^T$, $x \in \mathbb{H}$ and $y \in \{0, 1\}$ such that $z = \dfrac{x}{y}$, it follows that $\overline{z} = \dfrac{\overline{x}}{y}$.

Theorem 28. Given arbitrary $z, w \in \mathbb{H}^T$ it follows that:

a) $\overline{\overline{z}} = z$.

b) $\overline{z + w} = \overline{z} + \overline{w}$.

c) $\overline{-w} = -\overline{w}$.

d) $\overline{z - w} = \overline{z} - \overline{w}$.

e) $\overline{zw} = \overline{w}\ \overline{z}$.

f) $\overline{w^{-1}} = \overline{w}^{-1}$.

g) $\overline{z/w} = \overline{w} \backslash \overline{z}$ and $\overline{w \backslash z} = \overline{z}/\overline{w}$.

Proof. Let $z, w \in \mathbb{H}^T$ be arbitrary. Let $x, u \in \mathbb{H}$ and $y, t \in \{0, 1\}$ such that $z = \dfrac{x}{y}$ and $w = \dfrac{u}{t}$.

a) $\overline{\overline{z}} = \overline{\overline{\left(\frac{x}{y}\right)}} = \overline{\left(\frac{\overline{x}}{y}\right)} = \frac{\overline{\overline{x}}}{y} = \frac{x}{y} = z.$

b) If $y = t = 0$, $x \neq 0$ and $u \neq 0$ then $\overline{z+w} = \overline{\frac{x}{0} + \frac{u}{0}} = \overline{\left(\frac{\frac{x}{|x|} + \frac{u}{|u|}}{0}\right)} = \frac{\overline{\frac{x}{|x|} + \frac{u}{|u|}}}{0} = \frac{\overline{\left(\frac{x}{|x|}\right)} + \overline{\left(\frac{u}{|u|}\right)}}{0} = \frac{\frac{\overline{x}}{\overline{|x|}} + \frac{\overline{u}}{\overline{|u|}}}{0} = \frac{\frac{\overline{x}}{|\overline{x}|} + \frac{\overline{u}}{|\overline{u}|}}{0} = \frac{\overline{x}}{0} + \frac{\overline{u}}{0} = \overline{\left(\frac{x}{0}\right)} + \overline{\left(\frac{u}{0}\right)} = \overline{z} + \overline{w}$. Otherwise $\overline{z+w} = \overline{\frac{x}{y} + \frac{u}{t}} = \overline{\left(\frac{xt+uy}{yt}\right)} = \frac{\overline{xt+uy}}{yt} = \frac{\overline{x}\,\overline{t} + \overline{u}\,\overline{y}}{yt} = \frac{\overline{x}t + \overline{u}y}{yt} = \frac{\overline{x}}{y} + \frac{\overline{u}}{t} = \overline{\left(\frac{x}{y}\right)} + \overline{\left(\frac{u}{t}\right)} = \overline{z} + \overline{w}.$

c) $\overline{-w} = \overline{\left(-\frac{u}{t}\right)} = \overline{\left(\frac{-u}{t}\right)} = \frac{\overline{-u}}{t} = \frac{-\overline{u}}{t} = -\frac{\overline{u}}{t} = -\overline{\left(\frac{u}{t}\right)} = -\overline{w}.$

d) It follows from (b) and (c).

e) $\overline{zw} = \overline{\left(\frac{x}{y}\frac{u}{t}\right)} = \overline{\left(\frac{xu}{yt}\right)} = \frac{\overline{xu}}{yt} = \frac{\overline{u}\,\overline{x}}{yt} = \frac{\overline{u}\,\overline{x}}{ty} = \frac{\overline{u}}{t}\frac{\overline{x}}{y} = \overline{\left(\frac{u}{t}\right)}\,\overline{\left(\frac{x}{y}\right)} = \overline{w}\,\overline{z}.$

f) If $u \neq 0$ then $\overline{w^{-1}} = \overline{\left(\frac{u}{t}\right)^{-1}} = \overline{\left(\frac{\frac{t}{u}}{1}\right)} = \frac{\overline{\left(\frac{t}{u}\right)}}{1} = \frac{\frac{\overline{t}}{\overline{u}}}{1} = \left(\frac{\frac{\overline{t}}{\overline{u}}}{1}\right) = \left(\frac{\overline{u}}{\overline{t}}\right)^{-1} = \overline{\left(\frac{u}{t}\right)}^{-1} = \overline{w}^{-1}$. Otherwise $\overline{w^{-1}} = \overline{\left(\frac{u}{t}\right)^{-1}} = \overline{\left(\frac{t}{u}\right)} = \frac{\overline{t}}{\overline{u}} = \frac{t}{u} = \left(\frac{u}{t}\right)^{-1} = \left(\frac{\overline{u}}{\overline{t}}\right)^{-1} = \overline{\left(\frac{u}{t}\right)}^{-1} = \overline{w}^{-1}.$

g) It follows from items (e) and (f).

$\square$

Definition 29. Given $z \in \mathbb{H}^T$, take $x \in \mathbb{H}$ and $y \in \{0,1\}$, such that $z = \frac{x}{y}$, and define $|z| := \frac{|x|}{|y|}$. We call $|z|$ the *modulus* of the transquaternion z.

Once more we are abusing notation when we reuse the symbol for modulus. However, again, when we say that $|z| = \frac{|x|}{|y|}$ it is clear that the symbol "$|\ \ |$" on the left hand side of the equality refers to the modulus in $\mathbb{H}^T$, while the symbol "$|\ \ |$", on the right hand side of the equality, refers to the norm in $\mathbb{H}$. When the operation of taking the modulus in $\mathbb{H}^T$ is restricted to $\mathbb{H}$ it coincides with the norm on $\mathbb{H}$.

Theorem 30. The modulus of a transquaternion is well defined. That is, the modulus is independent of the choice of the fraction which represents the transquaternion. In other words, if $x, w \in \mathbb{H}$ and $y, t \in \{0,1\}$ and $\frac{x}{y} = \frac{w}{t}$ then $\left|\frac{x}{y}\right| = \left|\frac{w}{t}\right|$.

Proof. Let $x, w \in \mathbb{H}$ and $y, t \in \{0,1\}$ such that $\frac{x}{y} = \frac{w}{t}$. If $y = 1$ then $t = 1$, whence $x = w$ and the result is immediate. If $y = 0$ and $x = 0$ then

$t=0$ and $w=0$, whence the result is also immediate. If $y=0$ and $x\neq 0$ then $t=0$ and $w\neq 0$, whence $\left|\frac{x}{y}\right| = \infty = \left|\frac{w}{t}\right|$. $\square$

Again, let $z\in\mathbb{H}^T$, $x\in\mathbb{H}$ and $y\in\{0,1\}$, such that $z=\frac{x}{y}$. It follows that $|z|=\frac{|x|}{y}$.

Theorem 31. Given arbitrary $z,w\in\mathbb{H}^T$ it follows that:

a) $|z|^2 = z\overline{z}$.

b) $|\overline{z}| = |z|$.

c) $|zw| = |z||w|$.

d) $|w^{-1}| = |w|^{-1}$.

e) $|z/w| = |z|/|w| = |w|\backslash|z| = |w\backslash z|$.

f) $|z+w| \not> |z|+|w|$.

Proof. Consider arbitrary $z,w\in\mathbb{H}^T$. Suppose $z=\frac{x}{y}$ and $w=\frac{u}{t}$ where $x,u\in\mathbb{H}$ and $y,t\in\{0,1\}$.

a) $z\overline{z} = \frac{x}{y}\overline{\left(\frac{x}{y}\right)} = \frac{x}{y}\frac{\overline{x}}{y} = \frac{x\overline{x}}{y^2} = \frac{x\overline{x}}{y^2} = \frac{|x|^2}{y^2} = \frac{|x|}{y}\frac{|x|}{y} = |z||z| = |z|^2$.

b) $|\overline{z}| = \left|\overline{\left(\frac{x}{y}\right)}\right| = \left|\frac{\overline{x}}{y}\right| = \frac{|\overline{x}|}{y} = \frac{|x|}{y} = |z|$.

c) $|zw| = \left|\frac{x}{y}\frac{u}{t}\right| = \left|\frac{xu}{yt}\right| = \frac{|xu|}{yt}$
$= \frac{|x||u|}{yt} = \frac{|x|}{y}\frac{|u|}{t} = |z||w|$.

d) If $u\neq 0$ then $|w^{-1}| = \left|\left(\frac{u}{t}\right)^{-1}\right| = \left|\frac{\frac{t}{u}}{1}\right| = \frac{\left|\frac{t}{u}\right|}{1} = \frac{\frac{|t|}{|u|}}{1} = \left(\frac{|u|}{|t|}\right)^{-1} = \left|\frac{u}{t}\right|^{-1} = |w|^{-1}$. Otherwise, $|w^{-1}| = \left|\left(\frac{u}{t}\right)^{-1}\right| = \left|\frac{t}{u}\right| = \frac{|t|}{|u|} = \left(\frac{|u|}{|t|}\right)^{-1} = \left|\frac{u}{t}\right|^{-1} = |w|^{-1}$.

e) It follows from (c) and (d).

f) (I) If $z=\Phi$ or $w=\Phi$, say $z=\Phi$, then $|z+w| = |\Phi+w| = |\Phi| = \Phi \not> \Phi = \Phi+|w| = |\Phi|+|w| = |z|+|w|$.
(II) If $z\in\mathbb{H}$ and $w\in\mathbb{H}$ the result follows from the ordinary Triangle Inequality of quaternions.
(III) If either $z\in\mathbb{H}$ and $w\in\mathbb{H}^T_\infty$ or $z\in\mathbb{H}^T_\infty$ and $w\in\mathbb{H}$, say $z\in\mathbb{H}$ and $w\in\mathbb{H}^T_\infty$, then $z+w\in\mathbb{H}^T_\infty$, whence $|z+w| = \infty \not> \infty = |z|+\infty = |z|+|w|$.
(IV) If $z,w\in\mathbb{H}^T_\infty$ and $z\neq -w$ then $z+w\in\mathbb{H}^T_\infty$, whence $|z+w| = \infty \not> \infty = \infty+\infty = |z|+|w|$.
(V) If $z,w\in\mathbb{H}^T_\infty$ and $z=-w$ then $z+w=\Phi$, whence $|z+w| = |\Phi| = \Phi \not> \infty = \infty+\infty = |z|+|w|$.

$\square$

7 Transquaternion Metric and Topology

Let $D := \{z \in \mathbb{H};\ |z| < 1\}$, $\overline{D} := \{z \in \mathbb{H};\ |z| \leq 1\}$ and

$$\begin{array}{rcl} \varphi : \mathbb{H}^T \setminus \{\Phi\} & \to & \overline{D} \subset \mathbb{H}^T \\ r\mathcal{A}(\theta, \phi, \psi) & \mapsto & \dfrac{1}{1+\frac{1}{r}}\mathcal{A}(\theta, \phi, \psi) \end{array}.$$

Note that $\varphi_{|\mathbb{H}}$ is an homeomorphism between $\mathbb{H}$ and D with respect to the usual topology on $\mathbb{H}$, the topology induced by the norm of $\mathbb{H}$.

Theorem 32. Define $d : \mathbb{H}^T \times \mathbb{H}^T \to \mathbb{R}$ where

$$d(z, w) = \begin{cases} 0, & \text{if } z = w = \Phi \\ 2, & \text{if } z = \Phi \text{ or else } w = \Phi. \\ |\varphi(z) - \varphi(w)|, & \text{otherwise} \end{cases}$$

It follows that d is a metric on $\mathbb{H}^T$ and, therefore, $\mathbb{H}^T$ is a metric space.

Proof. Clearly, for all $z, w \in \mathbb{H}^T$, $d(z, w) = 0$ if and only if $z = w$, $d(z, w) = d(w, z)$ and $d(z, w) \geq 0$. If z, w and u are all non nullity then $d(z, u) = |\varphi(z) - \varphi(u)| = |\varphi(z) - \varphi(w) + \varphi(w) - \varphi(u)| \leq |\varphi(z) - \varphi(w)| + |\varphi(w) - \varphi(u)| = d(z, w) + d(w, u)$. The reader can verify that the Triangle Inequality is also true when at least one among z, w and u is nullity. $\square$

Theorem 33. The topology on $\mathbb{H}$ induced by the metric topology of $\mathbb{H}^T$ is the usual topology of $\mathbb{H}$. That is, if $U \subset \mathbb{H}^T$ is open on $\mathbb{H}^T$ then $U \cap \mathbb{H}$ is open (in the usual sense) on $\mathbb{H}$ and if $U \subset \mathbb{H}$ is open (in the usual sense) on $\mathbb{H}$ then U is open on $\mathbb{H}^T$.

Proof. Let us denote, for all $z \in \mathbb{H}^T$ and all positive $\rho \in \mathbb{R}$, the ball of centre z and radius ρ on $\mathbb{H}^T$ as $B_{\mathbb{H}^T}(z, \rho)$, that is, $B_{\mathbb{H}^T}(z, \rho) = \{w \in \mathbb{H}^T;\ |\varphi(z) - \varphi(w)| < \rho\}$, and, for all $z \in \mathbb{H}$ and all positive $\rho \in \mathbb{R}$, the ball of centre z and radius ρ on $\mathbb{H}$ as $B_{\mathbb{H}}(z, \rho)$, that is, $B_{\mathbb{H}}(z, \rho) = \{w \in \mathbb{H};\ |z - w| < \rho\}$.

Let $U \subset \mathbb{H}^T$ be open on $\mathbb{H}^T$ and let $z \in U \cap \mathbb{H}$. As U is open on $\mathbb{H}^T$, there is a positive $\varepsilon \in \mathbb{R}$ such that $B_{\mathbb{H}^T}(z, \varepsilon) \subset U$. As $\varphi_{|\mathbb{H}}$ is continuous, there is a positive $\delta \in \mathbb{R}$ such that if $w \in \mathbb{H}$ and $|z - w| < \delta$ then $|\varphi(z) - \varphi(w)| < \varepsilon$. Thus $B_{\mathbb{H}}(z, \delta) \subset B_{\mathbb{H}^T}(z, \varepsilon) \cap \mathbb{H} \subset U \cap \mathbb{H}$, whence $U \cap \mathbb{H}$ is open (in the usual sense) on $\mathbb{H}$.

Now, let $U \subset \mathbb{H}$ be open (in the usual sense) on $\mathbb{H}$ and let $z \in U$. Notice that there are $r \in [0, \infty)$ and $\theta, \phi \in [0, \pi]$ and $\psi \in (-\pi, \pi]$ such that $z = r\mathcal{A}(\theta, \phi, \psi)$. As U is open (in the usual sense) on $\mathbb{H}$, there is a positive $\varepsilon \in \mathbb{R}$ such that $B_{\mathbb{H}}(z, \varepsilon) \subset U$. As $\varphi^{-1}_{|D}$ is continuous, there is a positive $\delta \in \mathbb{R}$ such that $\delta < |\varphi(z) - \mathcal{A}(\theta, \phi, \psi)|$ and if $\varphi(w) \in D$ and $|\varphi(z) - \varphi(w)| < \delta$ then $|z - w| < \varepsilon$. Since $\varphi(w) \in D$ it follows that $w \in \mathbb{H}$. Thus $B_{\mathbb{H}^T}(z, \delta) \subset B_{\mathbb{H}}(z, \varepsilon) \subset U$, whence U is open on $\mathbb{H}^T$. $\square$

Theorem 34. φ is an homeomorphism.

Proof. Clearly φ is bijective.

Let $z \in \mathbb{H}^T \setminus \{\Phi\}$ be arbitrary. Let $\varepsilon \in \mathbb{R}$ be positive arbitrary. By Theorem 33, $B_{\mathbb{H}^T}(\varphi(z), \varepsilon) \cap \mathbb{H}$ is open in the usual sense on $\mathbb{H}$, whence there is a positive $\delta \in \mathbb{R}$ such that $B_{\mathbb{H}}(\varphi(z), \delta) \subset B_{\mathbb{H}^T}(\varphi(z), \varepsilon) \cap \mathbb{H}$. Thus if $w \in \mathbb{H}^T$ and $d(w, z) < \delta$ then $|\varphi(w) - \varphi(z)| < \delta$, whence $\varphi(w) \in B_{\mathbb{H}}(\varphi(z), \delta)$ which implies $\varphi(w) \in B_{\mathbb{H}^T}(\varphi(z), \varepsilon) \cap \mathbb{H}$ and, thereby, $d(\varphi(w), \varphi(z)) < \varepsilon$. Therefore φ is continuous at z.

In a similar way we see that φ^{-1} is also continuous. □

Remark 35. Because of Theorem 33:

i) Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{H}$ and let $L \in \mathbb{H}$, it follows that $\lim\limits_{n\to\infty} x_n = L$ on $\mathbb{H}^T$ if and only if $\lim\limits_{n\to\infty} x_n = L$, in the usual, sense on $\mathbb{H}$.

ii) Let $A \subset \mathbb{H}$, $f : A \to \mathbb{H}$, $x \in A'$ and $L \in \mathbb{H}$, it follows that $\lim\limits_{x\to x} f(z) = L$ on $\mathbb{H}^T$ if and only if $\lim\limits_{x\to x} f(z) = L$, in the usual sense, on $\mathbb{H}$.

iii) Given $x \in A$, it follows that f is continuous in x on $\mathbb{H}^T$ if and only if f is continuous in x, in the usual sense, on $\mathbb{H}$.

Theorem 36. $\mathbb{H}^T$ is disconnected.

Proof. $\mathbb{H}^T = \left(\mathbb{H} \cup \mathbb{H}^T_\infty\right) \cup \{\Phi\}$ and the sets $\mathbb{H} \cup \mathbb{H}^T_\infty$ and $\{\Phi\}$ are open. □

Notice that Φ is the unique isolated point of $\mathbb{H}^T$.

Remark 37. Since Φ is an isolated point of $\mathbb{H}^T$:

i) Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{H}^T$. Notice that $\lim\limits_{n\to\infty} x_n = \Phi$ if and only if there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$.

ii) Let $A \subset \mathbb{H}$, $f : A \to \mathbb{H}^T$ and $x \in A'$, it follows that $\lim\limits_{x\to x} f(z) = \Phi$ if and only if there is a neighbourhood U of x such that $f(z) = \Phi$ for all $x \in U \setminus \{x\}$.

iii) If $\Phi \in A$ then f is continuous in Φ.

Theorem 38. $\mathbb{H}^T$ is a separable space.

Proof. $(\mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q}) \cup \{\Phi\}$ is countable and dense in $\mathbb{H}^T$. □

Theorem 39. $\mathbb{H}^T$ is compact.

Proof. Since $\overline{D}$ is compact under the usual topology, by Theorem 33, $\overline{D}$ is compact under the topology of $\mathbb{H}^T$. Since φ is an homeomorphism and $\overline{D}$ is compact it follows that $\mathbb{H}^T \setminus \{\Phi\}$ is compact. Thus $(\mathbb{H}^T \setminus \{\Phi\}) \cup \{\Phi\}$ is compact. □

Theorem 40. $\mathbb{H}^T$ is complete.

Proof. Every compact metric space is complete. □

Since $\mathbb{H}^T$ is separable compact complete metric topological space, $\mathbb{H}^T$ has all properties of such a space.

8 Discussion

The quaternions are used in various applications in computer science and physics. The transquaternions obviously have an application as an exception-free version of the quaternions in computing but it would be interesting to know if they have applications in physics, developing the use of transreal numbers in physical equations [4]? Quaternions have been used to describe all four of Maxwell's equations in a single differential equation of a quaternion variable. We wonder if Maxwell's equations can be totalised and, if they can, do the singularities of each individual equation coincide with the transquaternion singularities? We would also like to know if empirically observable singularities in electrodynamics can be analysed by the putative trans-Maxwell's-equations in transvector ([2], Definition 3.2) or transquaternion form?

There are four normed division algebras based on the field of real numbers. These are the reals, complexes, quaternions, and octonions. The first three of these have been totalised, *ab initio,* as the transreals [1] [6], transcomplexes [3] [5] and, now, the transquaternions. All four of the usual normed division algebras can be developed using the Cayley-Dickson Construction but this construction relies on the Cartesian form of complex numbers, which is degenerate for transnumbers, where a polar form must be used. Nonetheless we wonder if a polar form of this construction can be developed that constructs the transnumber systems, similar to transfields ([6], Section IV).

The four normed division algebras are the first four algebras in a sequence of algebras generated by the Cayley-Dickson Construction. Real algebra appears as the zero'th algebra in the sequence, it operates on a $2^0 = 1$ tuple of real numbers. Complex algebra appears as the first algebra, counting from zero, and operates on a $2^1 = 2$ tuple of real numbers. Quaternion algebra is the second algebra, counting from zero, and operates on a $2^2 = 4$ tuple of real numbers. Octonion algebra is the third algebra, counting from zero, and operates on a $2^3 = 8$ tuple of real numbers. Further algebras are generated in sequence. Thus the n'th algebra operates on a tuple of 2^n real numbers. We wonder if every non-negative integer, n, generates a transalgebra operating on a tuple of 2^n transreal numbers? In transreal arithmetic, there is a greatest transreal number, $2^\infty = \infty$. We wonder if there is a last, infinity'th, transalgebra generated by the putative polar form of the Cayley-Dickson Construction? Is there a nullity'th transalgebra operating on a tuple of $2^\Phi = \Phi$ transreal numbers? What values can n take in the construction of algebras operating on a 2^n tuple of transreal numbers?

We can choose a metric so that each of the transreal, transcomplex and transquaternion algebras are homeomorphic to a unit ball, together with an isolated point at nullity. Thus the ordered transreal numbers correspond to a 1-ball whose surface is a 0-sphere. The interior of the ball is a line segment, which is bijective with the real number line, while the sphere is composed of the two end-points terminating the line segment, which are bijective with the transreal positive infinity and negative infinity. Similarly the transcomplex numbers are a 2-ball whose surface is a 1-sphere. The interior of the ball is bijective with the polar-complex disc and with the Cartesian-complex-plane. The sphere is the circle at infinity. Finally, the transquaternions correspond to a 4-ball whose surface is a 3-sphere. The interior of the ball is bijective with the space of quaternions and the sphere is the sphere at infinity. In every case, nullity corresponds to an isolated point that lies outside the space in which the ball is embedded. We wonder if transoctonions correspond to an 8-ball whose surface is a 7-sphere, together with an isolated point at nullity? More generally, we wonder whether every n'th transalgebra corresponds to a 2^n ball whose surface is a 2^{n-1} ball, together with an isolated point at nullity?

9 Conclusion

We introduce transquaternions, which totalise the arithmetical operations of quaternion addition, subtraction, multiplication, and both left and right division. In particular, division of quaternions by zero is allowed.

Transquaternions are a separable compact complete metric topological space. The transquaternions are homeomorphic to the unit hypersphere or glome including its interior, together with an isolated point at nullity. The usual quaternions fill out the 4D interior of the hypersphere. The infinite transquaternions are produced by dividing a non-zero quaternion by zero. They fill out the 3D surface of the hypersphere. The nullity transquaternion is produced by dividing zero by zero – it is an isolated point that lies outside the 4D space containing the hypersphere.

We propose future work, including the application of transnumbers in physics, the development of the transoctonions, and an examination of the construction of Cayley-Dickson algebras.

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