Foundations of Transmathematics

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Agenda

- Motivation - total foundation for mathematics
- Von Neumann Ordinals
- Russell's Paradox Dissolved
- Nullity - the unordered number as a pure set
- Infinity - as the Russell Set, excluding nullity
- Transordinals
- Paraconsistency - naive set theory is a paraconsistent logic
- Conclusion

Motivation

- Transmathematics aims to develop total systems
- Transmathematics has been developed in the usual set theory ZFC
- ZFC is easy to use but:
- ZFC has no set big enough to be infinity
- ZFC is partial

Motivation

- NFU has a Universal Set that is big enough to be infinity but:
- NFU is difficult to use because it uses a type system to avoid Russell's Paradox

Motivation

- Naive set theory has a Universal Set and is easy to use but:
- It is said to be incoherent because of Russell's Paradox
- So let's dissolve Russell's Paradox by showing that naive set theory is a paraconsistent logic!

Motivation

- If we can construct the transordinals in a set theory then this set theory is a sufficient basis for transmathematics and the usual mathematics
- We construct the transordinals in naive set theory

Von Neumann Ordinals

- $0 = \{\}$
- $1 = \{0\}$
- $2 = \{0,1\}$
- $3 = \{0,1,2\}$
- No von Neumann ordinal contains itself so every von Neumann ordinal is a member of the Russell Set
- Name the set of all ordinals $\mathcal{O}$

Russell's Paradox

- Use set-builder notation
- The Russell Set is $R_s = \{x_1 \mid x_2 \notin x_3\}$ where $x_1 = x_2 = x_3$
- The Russell Element is $R_e = x_1$ where $R_e = R_s$
- Use lazy evaluation
- Now $x_1 = 0$ is in R_s because $x_2 = 0 \notin x_3 = 0$
- And similarly for all von Neumann Ordinals!

Russell's Paradox

- As usual $R_e \in R_s \iff R_e \notin R_s$ is a paradoxical bi-implication
- Mathematics says that R_s does not exist
- But transmathematics says it does - everything exists in a total system!
- In particular R_s contains all of the von Neumann ordinals but does it contain R_e?

Russell's Paradox

- Can we choose $x_1 = R_e$?
- Choices $R_e \in R_s = T$, $R_e \in R_s = F$ are both paradoxical
- But suppose the contradiction $R_e \in R_s = TF$
- Choose $R_e \in R_s = T \implies R_e \notin R_s = F$ and
- $R_e \in R_s = F \implies R_e \notin R_s = T$
- This satisfies the paradox in both directions but R_e is a contradictory object that is both in and out of R_s

Russell's Paradox

- Contradictions are forbidden by The Law of the Excluded Middle, therefore we cannot choose $x_1 = R_e$
- But we can choose $x_1 \neq R_e$, for example, x_1 is any von Neumann ordinal
- Therefore R_s exists and, unequivocally, $R_e \notin R_s$

Russell's Paradox

- Is there some irreparable error in our reasoning?
- Is there a genuine choice between saying that R_e does not exist, as we have done, or R_s does not exist, as the usual mathematics does?
- Is the usual mathematics wrong?

Russell's Paradox

- Re-write the set $R_s = \{x_1 \mid x_2 \notin x_3\}$
- As Prolog predicate $InR_s(x_1) \vdash x_2 \notin x_3$
- Binding R_e to x_1 gives $R_e \in R_s = R_e \notin R_s = F$
- Binding R_e to x_2 and R_s to x_3 gives $R_e \notin R_s = R_e \in R_s = F$
- Prolog binding dissolves Russell's paradox
- Prolog binding is equality
- We can easily define set theories with equality!

Russell’s Paradox

- As I proposed in the first transmathematica conference:
- Re-write $\{x \mid \phi(x)\}$ as $(x = x)\ \&\ \phi(x)$
- This total set theory blocks Russell’s Paradox and contains all set theories
- The class of all classes is partitioned into the Universal Set's interior and exterior
- The exterior contains any atoms, antinomies, physical objects or anything else that is not in the interior

Nullity

- The von Neumann ordinals are ordered by membership $x < y \iff x \in y$
- Define nullity as $\Phi = \{\{\{\}\}\}$
- Then nullity is the simplest set that is unordered with respect to the von Neumann ordinals
- If we use a different model of the ordinals we may have to use a different model of nullity

Infinity

- Define $\infty = R_s \backslash \{\Phi\}$, now
- Nullity is unordered with respect to infinity
- All von Neumann ordinals are less than infinity
- Infinity is the greatest ordinal

Transordinals

- Define transordinals as $\mathscr{O}^T = \mathscr{O} \cup \{\infty\} \cup \{\Phi\}$
- Nullity is the uniquely unordered transordinal
- Infinity is the greatest of the ordered transordinals

Paraconsistency

- The Russell Paradox introduces a contradiction to mathematics
- But mathematics does not blow up (allow all possible theorems to follow from a contradiction)
- Sophisticated logics and set theories use types to avoid this contradiction

Paraconsistency

- Equality is enough to dissolve Russell's Paradox
- All logics and set theories, including naive set theory, say contradictory objects do not exist
- This removes contradictions from the domain of discourse so all logics and set theories do not blow up - they are all paraconsistent
- The usual mathematics is paraconsistent

Conclusion

- When the ordinals, $\mathcal{O}$, are modelled by the von Neumann ordinals
- Nullity is the simplest unordered set $\Phi = \{\{\{\}\}\}$
- Infinity is the Russell Set, excluding nullity, $\infty = R_s \backslash \{\Phi\}$
- The transordinals are $\mathcal{O}^T = \mathcal{O} \cup \{\infty\} \cup \{\Phi\}$

Conclusion

- When naive set theory uses $\{x \mid \phi(x)\}$ as a shorthand for $(x = x)\ \&\ \phi(x)$ then Russell's Paradox does not exist
- This set theory is paraconsistent, as all set theories are
- The class of all classes is partitioned into the Universal Set's interior and exterior
- The Universal Set's exterior contains atoms, antinomies, physcial objects and anything that is not in the interior
- I provisionally name this set theory FT - Foundations of Transmathematics