Division by Zero: A Survey of Options

Jan A. Bergstra
Informatics Institute, University of Amsterdam
Science Park, 904, 1098 XH, Amsterdam, The Netherlands
j.a.bergstra@uva.nl, janaldertb@gmail.com

Submitted: 12 May 2019
Revised: 24 June 2019

Abstract

The idea that, as opposed to the conventional viewpoint, division by zero may produce a meaningful result, is long standing and has attracted interest from many sides. We provide a survey of some options for defining an outcome for the application of division in case the second argument equals zero. The survey is limited by a combination of simplifying assumptions which are grouped together in the idea of a premeadow, which generalises the notion of an associative transfield.

1 Introduction

The number of options available for assigning a meaning to the expression $1/0$ is remarkably large. In order to provide an informative survey of such options some conditions may be imposed, thereby reducing the number of options. I will understand an option for division by zero as an arithmetical datatype, i.e. an algebra, with the following signature:

- a single sort with name V,
- constants 0 (zero) and 1 (one) for sort V,
- 2-place functions $\_ \cdot \_$ (multiplication) and $\_ + \_$ (addition),
- unary functions $-\_$ (additive inverse, also called opposite) and $\_^{-1}$ (multiplicative inverse),
- 2 place functions $\_ - \_$ (subtraction) and $\_/\_$ (division).

Decimal notations like $2, 17, -8$ are used as abbreviations, e.g. $2 = 1 + 1$, and $-3 = -((1+1)+1)$. With inverse the multiplicative inverse is meant, while the additive inverse is referred to as opposite.

This signature is referred to as the signature of meadows Σ_{Md} in [6], with the understanding that both inverse and division (and both opposite and subtraction) are present.

A number of assumptions will be made with the intention to limit the scope of the discussion to manageable proportions. We consider an arithmetic datatype with domain $|V|$ for sort V, and we assume that the following conditions are met.

Nontriviality. $0 \neq 1$.

Totality. Addition, opposite, subtraction, multiplication, inverse, and division are total functions.

Additive monoid. $(|V|, \_ + \_, 0)$ is a commutative monoid (for all $x \in |V|$, $x + 0 = 0 + x = x$, and addition is commutative and associative).

Multiplicative monoid. $(|V|, \_ \cdot \_, 1)$ is a commutative monoid (for all $x \in |V|$, $x \cdot 1 = 1 \cdot x = x$, and multiplication is commutative and associative).

Zero is idempotent. $0 \cdot 0 = 0$ (that 1 is idempotent follows from the additive monoid requirement.)

Opposite. $x+y = 0 \to y = -x$ (opposite produces all proper additive inverses).

The opposite of one is a proper additive inverse. $1 + (-1) = 0$.

Inverse. $x \cdot y = 1 \to y = x^{-1}$ (inverse produces all proper multiplicative inverses).

Subtraction. $x - y = x + (-y)$ (and therefore in combination with the additive monoidal structure: $-x = 0 - x$.)

Division. $x/y = x \cdot x^{-1}$ (and therefore in combination with the multiplicative monoidal structure: $x^{-1} = 1/x$.)

Arithmetical datatypes which meet these requirements may have many unusual properties. Below we will find that plausible equations like $x - x = 0, x \cdot x^{-1} = 1, x \cdot (y+z) = x \cdot y + x \cdot z, (1/x) \cdot (1/y) = 1/(x \cdot y)$ are refuted in some of the options for arithmetic that are considered. A (nontrivial) premeadow is a datatype (algebra) which satisfies each of the above requirements. We notice that unicity of proper additive inverses follows from the additive monoidal structure: if $x+y = 0 = x+z$, then $y = 0+y = (x+z)+y = (x+y)+z = 0+z = z$. Similarly the multiplicative monoidal structure implies that proper multiplicative inverses are unique.

The notion of a premeadow is very close to but slightly less demanding than the notion of a transfield as introduced in [21]. More precisely, all associative transfields are premeadows (a transfield may have a nonassicative addition). While a transfield must have a field as a substructure of its reduct to the signature of rings, a premeadow need not necessarily include a field. Every transfield is a premeadow, however, and every premeadow for which the reduct upon forgetting the inverse function and division contains a field as a substructure is a transfield. In particular every meadow is a premeadow, but not every meadow is a transfield. We refer to [12] for examples of meadows that are not transfields.

The constraints of a premeadow rule out certain mechanisms from floating point arithmetic. For instance if one assumes the existence of a positive zero, say 0^+ (which may or may not be identified with the unsigned zero 0) and a negative zero, say 0^-, where the negative zero satisfies $1 + 0^- = 1$ and $(-1) + 0^- = -1$ then it follows from the requirements of a premeadow that $0^- = 0^- + 0 = 0^- + (1 + (-1)) = (0^- + 1) + (-1) = (1 + 0^-) + (-1) = 1 + (-1) = 0$ and formalisation in terms of a premeadow cannot be used to explain how and why 0 and 0^- differ. The question to which extent existing as well as conceivable floating point models can be faithfully and productively modelled by means of algebras and abstract datatypes seems not to have received much attention and it seems fair to say that floating point arithmetic cannot be used as an argument in favour of any design of a specific arithmetical datatype involving division by zero.

Our discussion will be confined to premeadows with characteristic 0. Generalisation of the definitions below to the case of characteristic $p > 0$ is unproblematic, however.

Because subtraction and division are found by means of explicit definition from opposite and inverse, the presence of these 2 place operations is merely a matter of notational convenience below. The notion of a fraction, however, depends on the presence of a division operation in the signature. Following [8] a fraction is defined as expression over the signature $\Sigma_M d$, or an extension thereof, with division as the leading function symbol. For a fraction P/Q, P is called its numerator and Q is called its denominator. It follows from these conventions that both the numerator and the denominator of a fraction are expressions.

Definition 1.1. *A premeadow is an algebra with signature Σ_{Md} which satisfies the above 10 requirements with the possible exception of nontriviality. A nontrivial premeadow is a premeadow in which $0 \neq 1$.*

Premeadows constitute a quasi-variety because all of the listed requirements take the form of conditional equations.

1.1 On the significance of signatures for "division by zero"

The importance of being explicit about signatures is twofold. First of all with a signature comes a natural notion of syntax, and as a consequence a distinction between syntax and semantics. Secondly, upon having adopted the signature of meadows, the status of an expression $1/0$ is entirely unproblematic, and for that reason the question about its meaning is not only a reasonable issue, but it has even become an unavoidable matter. Working with signatures and in particular working with the signature of meadows creates a context in which the topic of division by zero arises in a natural manner. Questions about division by zero can only arise if inverse and/or division are understood as operations for which the interpretation is not fixed for once and for all. The situation for opposite and for inverse is comparable: the opposite of a value may not be a proper additive inverse of it, and the inverse of a value may not be a proper multiplicative inverse of it. In both cases we choose not to have a function symbol for "the proper inverse of" in the signature, and this choice is primarily motivated by the advantages of working with total algebras.

It is left up to the design of a model, that is the choice of semantics, to determine the meaning of any expression, in particular of expressions like $1/0$.

Σ_{Md} extends the signature of rings with inverse and division. The signature of fields, however, if one wishes to think in those terms, is the same as the signature of rings, and does not include either inverse or division. As a consequence, in the setting of fields, it is implausible to view, say the equation $x^{-1} \cdot y^{-1} = (x \cdot y)^{-1}$ as a potential axiom about the inverse function. Assuming in the context of fields that one considers inverse to be a partial function, then the meaning of this equation is quite sensitive to the logic of partial functions which one prefers to adopt. Indeed, adopting the axiom may have an impact on the domain of the inverse function, which introduces a risk of circular reasoning.

In the context of premeadows, however, such an equation is perfectly plausible as an axiom. The limitation to commutative multiplication is a matter of convenience, and most of the topics discussed in the paper admit generalisation to the non-commutative (also called skew) case.

The terminology of inverse functions merits some further attention. In the following definitions ϕ may be a partial function.

Definition 1.2. *A total or partial function $\phi(\_)$ captures multiplicative inverses if for all x and y in $|V|$, if $x \cdot y = 1$ then in $\phi(x)$ is defined and $x \cdot \phi(x) = 1$.*

Definition 1.3. *A unary operator $\phi(\_)$ is a multiplicative inverse operator if it captures multiplicative inverses.*

These definitions motivate the terminology used above when referring to $\_^{-1}$ as a multiplicative inverse on the basis of the requirement $x \cdot y = 1 \rightarrow y = x^{-1}$. It is possible that besides $\_^{-1}$ other functions, which may have an explicit definition over the signature of meadows, also capture multiplicative inverses. With inverse we will refer to the specific (total) multiplicative inverse $\_^{-1}$.

Definition 1.4. *A function $\phi(\_)$ produces multiplicative inverses if for all x in its domain, $x \cdot \phi(x) = 1$.*

From these assumptions an important conclusion can be derived:

Proposition 1.1. *The inverse operator in a non-trivial premeadow does not produce a multiplicative inverse for 0.*

Proof. Assume that $\_^{-1}$ produces a multiplicative inverse $q = 0^{-1}$ on 0, then: $0 = 0 \cdot 1 = 0 \cdot (0 \cdot q) = (0 \cdot 0) \cdot q = 0 \cdot q = 1$, which contradicts the non-triviality assumption on the premeadow at hand. □

In a similar manner one finds that if an inverse operator produces multiplicative inverses it cannot have 0 as a result. Many authors experiment with $1/\infty = 0$ or variations thereof and the following result applies to such proposals.

Proposition 1.2. *If the inverse operator in a non-trivial premeadow produces proper multiplicative inverses on all nonzero arguments it does not produce value 0 on any nonzero argument.*

Proof. Assume that $\_^{-1}$ is an operator which produces proper multiplicative inverses and which takes value 0 on argument $q \neq 0$, i.e. $q^{-1} = 0$ and $q \cdot q^{-1} = 1$. Then $1 = 1 \cdot 1 = (q \cdot q^{-1}) \cdot (q \cdot q^{-1}) = (q \cdot 0) \cdot (q \cdot 0) = q \cdot (0 \cdot (q \cdot 0)) = q \cdot ((q \cdot 0) \cdot 0) = q \cdot (q \cdot (0 \cdot 0)) = (q \cdot q) \cdot (0 \cdot 0) = (q \cdot q) \cdot 0 = q \cdot (q \cdot 0) = q \cdot (0 \cdot q) = q \cdot 1 = q$. Now combining $1 = q$ with the assumption $q \cdot q^{-1} = 1$ yields $1 = q \cdot q^{-1} = 1 \cdot 0 = 0$ which contradicts the non-triviality requirement. $\square$

The subject of division by zero, under the constraints mentioned above, is about the design of versions of arithmetic which include multiplication and an inverse function which produces proper inverses when these exist and which is defined on 0.

It is an implicit assumption that arithmetic is not fixed for once and for all, in other words that different designs of arithmetic are possible and are worth investigation. At the same time arithmetic is merely an intuitive notion which grasps systems of numbers equipped with constants and operations from various signatures.

1.2 Options outside the quasivariety of premeadows

Several authors have proposed a perspective on division by zero which allows for $0 \cdot 0^{-1} = 1$. For instance in [16] $1/0 = \infty$ is adopted in combination with $1 \cdot \infty = 1$. Such proposals are significant, but seem to be technically less accessible than proposals which are compatible with the requirements of premeadows. With Proposition 1.1 the existence of a proper inverse of zero is ruled out in premeadows, however. As stated above the survey of this paper will stay within the confinements of premeadows and therefore none of the options which are discussed below satisfy the identity $0 \cdot 0^{-1} = 1$.

2 From the conventional approach to $1/0$ to motivating alternatives

It is often assumed that the ordinary conventions for dealing with the partial multiplicative inverse operator are well-known and self-evident. That may be the case but writing explicitly about such conventions is not entirely straightforward. It seems to be the case that notations like $1/0$ have no place in professional mathematics. Not only do such notations not occur, such notations do not occur for a good reason which is understood to be based on common background knowledge. It is the nature of this background knowledge which must be made explicit, and then upon having been made explicit it may or may not require further substantiation. Now the mere fact that $1/0$ has no meaning is no principled reason for the strong preference for not using $1/0$ as a notation in a professional text.

Consider the infinite sums $P_k = \Sigma_{n=k}^{\infty} 1/n$ for k a nonzero natural number. It is reasonable to state that $P_1 = 1 + P_2$ and it is also reasonable to say that P_1 diverges, which means that it is not defined and therefore that it has no numerical value. The idea that P_1 diverges is compatible with $P_1 = \infty$. With $Q_k = \Sigma_{n=k}^{\infty} (-1)^n \cdot n$, however, it is still reasonable to state that $Q_1 = -1 + Q_2$ while the Q_k exist even less than the P_k as the Q_k represent sums which don't even converge to ∞ or to $-\infty$.

Now consider the assertion that $1/0$ is undefined. It seems to be the case that the latter assertion is not acceptable as a constituent of a competent mathematical text. One may conclude that $1/0$ is to a lesser extent a fraction than that $\Sigma_{n=k}^{\infty} 1/n$ is a sum.

2.1 Principled (conventional) interpretation of the division sign

Mathematical conventions allow remarkable flexibility. Working in binary notation it is common to read 10 as the number two and the equation $1 + 1 = 10$ makes sense. And working modulo 3 it is known that $1 + 2 = 0$. Working in a non-standard model of the Peano axioms of arithmetic addition and multiplication work in unexpected ways. However, when it comes to $1/0$ there seems to be no semantic flexibility left as $1/0 = q$ can only hold if $0 \cdot q = 1$, which is problematic indeed.

Suppose one introduces an absorptive error value $\perp$ such that $1/0 = \perp$ and such that the error propagates through all operations. Then $\perp + (-\perp) = \perp$. It follows that the introduction of an error value calls into question the use of the opposite, for which one may expect that adopting the rule $x + (-x) = 0$ constitutes a very firm commitment, a commitment the deviation of which is just as problematic, or even more problematic that deviating from a commitment to $0 \cdot 0^{-1} = 1$. It follows from this observation that already adopting the existence of an error value as the result of division by zero would be incompatible with deeply rooted intuitions and conventions on the use of mathematical notations.

It may be the case that the conception that $1/0$ does not exist is the most rigidly held convention of all conventions regarding the use of elementary mathematical notation. Not only is $1/0$ non-existent but as an idea it embodies the very concept of mathematical non-existence, which might even be prior to the concept of existence in the same way as the idea of inconsistency may be prior to the idea of consistency. It is intuitively obvious that the concept of the non-existence of a conceivable entity is hardly communicated in a best possible manner by having a preferred notation for precisely that entity.

2.2 Context dependence of occurrences of division

For an expression P of the form Q/R to occur in a text, in a specific position, it is conventionally required that the text embeds said (specific) occurrence of P in a context which is such that from the assumptions imposed on the parameters that are available in the given context it can be inferred, either formally or informally, but in any case to the reader's satisfaction, that R is nonzero.

For instance the assertion "for all $x \neq 0$ it is the case that $x \cdot (1/x) = 1$" complies with the mentioned requirement on contexts: $1/x$ occurs in a context in which it has been assumed that x is nonzero. The mentioned assertion is therefore regarded as unproblematic form the perspective of division by zero.

In contrast with these observations the first order sentence χ defined by $\chi \equiv \forall x \in \mathbb{Q}\,(x \neq 0 \to x \cdot (1/x) = 1)$ is somehow troublesome. For χ to be true it must be the case that substituting 0 for x produces a true statement $[0/x]\chi \equiv 0 \neq 0 \to 0 \cdot (1/0) = 1$. Now for $[0/x]\chi$ to be considered a valid assertion each of its components must have some truth value, at least when adopting a classical two-valued logic. Thus either $0 \cdot (1/0) = 1$ is considered true or $0 \cdot (1/0) \neq 1$ is considered true but conventional mathematics commits to neither of these as $0 \cdot (1/0)$ is considered undefined.

For $\Phi \equiv \forall x \in \mathbb{Q}(x \neq 0 \to x \cdot (1/x) = 1)$ to be considered true some additional assumptions are required. Indeed a plurality of disparate strategies can be used to that end. We list some options:

1. a logic of partial functions may be used (there are many options for logics of partial functions) which incorporates a three valued logic, or

2. the occurrence of $1/0$ is considered to denote an error value in an ordinary error algebra so that $0 \cdot (1/0) \neq 1$ is satisfied, or

3. the universal quantifier is understood as an infinitary conjunction by reading Φ as $\bigwedge_{q \in \mathbb{Q}, q \neq 0}(q \cdot (1/q) = 1)$, with the effect that no subformula involving $1/0$ appears, or

4. the implication connective in $[0/x]\chi$ is understood in terms of a short circuit logic, in which the conclusion of the implication is only evaluated once (i.e. after) the condition has been found valid.

2.3 Syntax versus semantics: not a conventional intuition

Returning to the legitimacy of writing expressions like $1/0$ in conventional mathematical practice we notice that in conventional mathematics it is not even plausible to ask for the value of $1/0$ because there is no notion of syntax which entitles $1/0$ to the status of being asked questions about. In other words, an expression is a tool for denoting an entity, and whoever makes use of the expression is supposed to have sorted out in advance why doing so is realistic. But conventional mathematics is also pragmatic and liberal to some extent. When stating that the fraction $4/6$ is not simplified, reference is made to the fraction as an expression, rather than to the rational number which it denotes. Asking if the fraction $1/0$ has a value is likely to be countered with the objection that $1/0$ is not a fraction, just because it has no value.

The idea of a contrast between syntax and semantics, where syntax consists of expressions, to the extent that one may always ask whether or not an expression is legal, and do so in advance of the determination of its meaning, at least in principle, is not supported by conventional mathematical practice. It is commonplace, however, to ask for an expression that signifies a semantic entity, for instance the non-negative square root of 2 once its existence has been made plausible. Thus, to the extent that conventional mathematical practice supports a distinction between syntax and semantics, the latter serves as a tool whereas the former takes priority. In the case of the inverse of zero, this convention implies that by the mere use of an expression $1/0$ it is already acknowledged that, at least in principle, the author (speaker) knows what this expression may mean and is able to provide an explanation of that meaning.

As a consequence of the limited separation of syntax and semantics the use of axioms for the specification of operations is somehow limited too. However attractive it may seem to be from a formal perspective the idea that inverse or division is merely specified by one or more first order axioms is not very appealing to most mathematicians. As a consequence of this bias against the use of syntactic methods it is conceivable that less than optimal use has been made of specification techniques and term rewriting techniques in the context of arithmetic.

2.4 Motivation(s) for having inverse defined on zero

various approaches to the inverse of zero are motivated in different ways. Four motives for investigating a specific approach can be distinguished at least:

Curiosity driven motivation. One may investigate a certain approach without any claim that is is of either practical or philosophical value. For instance:

- In [7] the option that $1/0 = 1$ is investigated without any regard to whether or not there is an advantage in adopting that option.
- Incorporating the idea of the Riemann sphere into arithmetic. (The theory of wheels: [25, 14, 15], with subsequent work in [24].)
- Making progress on the first order theory of fields. (E.g. in [19, 22].)
- In [26] the idea is explored that $1/0 = (-1)!$. The paper contains a significant attempt to survey the history of division by zero, which, however contradicts the claim of [13] that the history of division starts with the ancient Greeks.

Instrumental motivation for specific objectives. A certain approach may be considered useful for pursuing specific objectives which are at first sight unrelated to division by zero. Among such objectives we mention:

- Exploiting calculational correspondence with formal reasoning patterns in geometry and analysis (E.g [20] upon adopting $1/0 = 0$.)
- Simplification of semantics and proof theory. ($1/0 = 0$ as assumed in various proof checking systems.)
- Making use of the framework of equational abstract datatype specifications for the investigation of specific arithmetical datatypes. For instance upon adopting $1/0 = 0$, work towards a theory of fractions is reported in [8]). For algebraic datatype specification we mention e.g. [17].

Moderate preference for a specific design. The idea that a certain approach to division by zero is preferable to all competing approaches, including the conventional idea of having division by zero undefined, serves as a motivation for some work. The development of Transmathematics (see e.g [21]) incorporates an attempt to rework parts of mathematics in a preferred design involving a notion of division by zero based on signed infinities.

Exposing a strong and definite preference. An author or group of authors may be convinced that adopting a certain approach to division by zero will be beneficial for the development of mathematics at large. This conviction comes with the idea that the approach at hand provides a superior perspective in comparison to other approaches, including conventional mathematics.

For instance in [20] and work cited in that paper, the respective authors express a strong preference for $1/0 = 0 = 0/0$. Remarkably, authors from the Institute of Reproducing Kernels claim that setting $0 = 0/0 = 1/0$ is a discovery which dates back to 2014 only ([18, 23]). This claim is to our perception refuted by the existence of [22]. The latter paper claims no priority for the idea that $1/0 = 0$ while making much progress regarding the logical analysis of that idea, to such an extent that its place in the history of the subject seems to be undisputable.

Admittedly in [22] no strong preference for $1/0 = 0$ is formulated. The history of expressing a strong preference for calculating with $1/0 = 0$ throughout mathematics may well have started in 2014 just as claimed in [18, 23].

3 A survey of minimal premeadows of characteristic zero

A datatype is minimal if it has no proper subalgebras. Below we will consider minimal arithmetical datatypes only, thereby leaving the reals for later work. We assume that the set $\mathbb{Q}$ of rational numbers is given and that 0, 1, addition, opposite, subtraction, and multiplication on $\mathbb{Q}$ are known. We refrain from adopting any particular construction of the rational numbers, we merely assume that these are given as a set with known interpretations for each closed expression $1/q$ where q denotes a nonzero integer. We will consider arithmetical datatypes with a domain that includes $\mathbb{Q}$ and may extend it with at most three pairwise different values outside $\mathbb{Q}$ taken from $\bot, +\infty, -\infty$. All functions from Σ_{Md} except inverse extend the interpretations of these functions on $\mathbb{Q}$ while inverse and division are constrained by finding a proper inverse or quotient if one exits in $\mathbb{Q}$. With $\mathbb{Q}_\star$ we refer to the arithmetical datatype of rationals in which inverse and division are partial functions. $\mathbb{Q}_\star$ is not a premeadow but it can be easily turned into a premeadow $\mathbb{Q}_p$ by taking for 0^{-1} any rational number p as a value. Below we will discuss in some detail $\mathbb{Q}_0$ and $\mathbb{Q}_1$.

3.1 Totalising division with an absorptive element

The simplest idea for developing a premeadow which includes the rational numbers is to introduce $\bot$ outside $\mathbb{Q}$ as a new value which serves as an absorptive element. We use $\bot$ which is the customary notation in the theory of abstract datatypes for such an element where it is often referred to as an error element. In transmathematics (see e.g. [2]) the notation Φ is preferred for an absorptive element, and the negative connotation of an error is deliberately avoided.

This strategy may be applied to any partial algebra. The value $\bot$ is supposed to propagate through all functions and to serve as the inverse of 0: $0^{-1} = \bot$ and for all x, $x + \bot = \bot + x = -\bot = x - \bot = \bot - x = x \cdot \bot = \bot \cdot x = \bot^{-1} = \bot/x = x/\bot = \bot$.

The structure thus obtained is denoted with $\mathbb{Q}_\bot$. From the perspective of abstract datatypes $\mathbb{Q}_\bot$ is a very plausible structure, but, we found almost no literature about this particular structure. In [10], $\mathbb{Q}_\bot$ is referred to as the common meadow of rational numbers, thereby emphasizing its proximity to a common understanding of partiality of division, while writing, **a** for additional element instead of $\bot$ in order to avoid any distraction potentially caused by the traditional negative connotations of $\bot$. In [10] the the equational theory of $\mathbb{Q}_\bot$ is studied in considerable detail, resulting in a completeness theorem restricted to the case of characteristic zero. In [9] the same structure appears as the initial algebra of so-called fracpairs. In [21] the real number version of $\mathbb{Q}_\bot$ occurs as example 19.

3.1.1 Semantic justification

If for a closed expression t $\mathbb{Q}_\bot \models t = 0$ then this identity is entirely trustworthy for any mathematician. The fact that inverse was made total can only make calculations deviate in the direction of non-rational values, i.e in the direction of $\bot$.

One may criticise $\mathbb{Q}_\bot$ for being pessimistic about the information which may be obtained from expressions involving division by zero.

3.1.2 Practical justification

We have no information about the practical use which has been made of this particular totalised arithmetic. The equational theory of $\mathbb{Q}_\perp$ is somewhat cumbersome and seems not to provide an attractive platform for the systematic development of further equational specifications, either initial or loose, of datatypes which incorporate functions and sorts tailored to specific applications. It is an advantage of $\mathbb{Q}_\perp$ that the unconditional fraction addition rule (UFAR) holds:

$$\frac{x}{y} + \frac{u}{v} = \frac{x \cdot v + y \cdot u}{y \cdot v}$$

3.2 Zero as the inverse of zero

Instead of taking a new value $\perp$ as the outcome of 0^{-1} one may take an existing value for instance $0^{-1} = 0$, thus obtaining the arithmetical; datatype $\mathbb{Q}_0$. This structure, and its first order and equational theory has been first studied in significant detail in [19] and [22]. An equational initial algebra specification for $\mathbb{Q}_0$ is given in [11]. For further work on the equational theory of $\mathbb{Q}_0$ and its counterparts for real and complex numbers we refer to [4, 5, 6, 7].

Adopting $0^{-1} = 0$ underlies [20] and many related papers. The same design decision has been adopted in various theorem provers.

3.2.1 Semantic justification

From a point of view of symmetry setting 0^{-1} to 0 is a reasonable, and even elegant way of completing the graph of the partial inverse function to a total one. We see no principled justification for, say, the following equation $\frac{1}{(1/0)-(1/0)} = 0$ in terms of the concept of division or with the help of analytic or asymptotic methods. However, one may easily get used to $1/0 = 0$ and the consequences thereof and avoid negative consequences of this assumption. The fact that a systematic axiomatic approach becomes feasible serves as a justification for adopting a seemingly arbitrary value of inverse on 0.

In $\mathbb{Q}_0$ UFAR fails: $1 = 1/0 + 1/1 \neq (1 \cdot 1 + 0 \cdot 1)/(0 \cdot 1) = 1/0 = 0$ and as a consequence the theory of fractions for $\mathbb{Q}_0$ is more involved than the fraction theory of $\mathbb{Q}_\perp$. Instead of UFAR the conditional fraction addition rule (CFAR) is valid in $\mathbb{Q}_0$:

$$x \neq 0 \wedge y \neq 0 \rightarrow \frac{x}{y} + \frac{u}{v} = \frac{x \cdot v + y \cdot u}{y \cdot v}$$

3.2.2 Practical justification

Four lines of practical justification can currently be distinguished: (i) based on the assumption that $0^{-1} = 0$ a rich meta-theory can be developed, (ii) we found that the equational theory of arithmetic with inverse is made total by adopting $0^{-1} = 0$ is attractive and allows for extension in several useful directions, (iii) by adopting $0^{-1} = 0$ the logic of proof checkers may be simplified in a useful manner, while preventing errors of calculation and proof is not made much harder, (iv) Some authors claim that adopting $0^{-1} = 0$ works unexpectedly well in various different branches of mathematics (see [20] and related papers).

3.3 One as the inverse of zero

Taking $0^{-1} = 1$ the structure $\mathbb{Q}_1$ is obtained. This option has been investigated in detail in [7]. We notice that $\mathbb{Q}_1$ satisfiers CFAR but not UFAR:

$$1 + 1 = \frac{1}{0} + \frac{1}{0} \neq \frac{1 \cdot 0 + 0 \cdot 1}{0 \cdot 0} = \frac{0 + 0}{0} = \frac{0}{0} = 0 \cdot 1 = 0$$

3.3.1 Justification

A semantic justification of this arithmetical datatype is comparable to the justification given for $\mathbb{Q}_0$ though somewhat less convincing because there is less symmetry, as inverse ceases to be an involution, and the equational theory becomes more involved and less attractive without any perceivable gains.

Apart from some spurious mentioning of the idea that $0^{-1} = 1$ in the educational literature we have not seen any systematic work on this basis. At this stage we are unaware of any practical justification for this design. One might just as well consider $0^{-1} = 731$ giving $\mathbb{Q}_{731}$ and agree in advance that no definition or theorem is made dependent of the parameter 731 which could just as well have been replaced by 732, or any other arbitrary rational number. Perhaps it is an advantage of working with $0^{-1} = 731$ in $\mathbb{Q}_{731}$ that it is easy to detect any unwanted dependency of a further result or development from the ad hoc number 731.

3.4 Unsigned ∞ as the inverse of zero: uncommon wheels

A different idea is to let 0^{-1} take a new value q outside $\mathbb{Q}$ (just as $\perp$) but in such a manner that $q^{-1} = 0$. It is customary to denote q with ∞ in this case. The simplest design of an arithmetical datatype of this kind, denoted with $\mathbb{Q}_\infty$ is as follows:

$\mathbb{Q}_\infty$ has domain $\mathbb{Q} \cup \{\infty\}$ and the operations are extended as follows (assuming that the resulting structure will be a premeadow):

- $0^{-1} = \infty$ and $\infty^{-1} = 0$,
- $\infty \cdot x = \infty + x = -\infty = \infty$.

3.4.1 Semantic justification

A semantic justification for $\mathbb{Q} \cup \{\infty\}$ may be as follows: some closed expressions which evaluate to $\perp$ in $\mathbb{Q}_\perp$ are now evaluated to a meaningful value, for instance:

$$1/(1/0) = \lim_{x \downarrow 0} 1/(1/x) = \lim_{x \uparrow 0} 1/(1/x) = \lim_{x \to 0} x = 0$$

However a similar motivation can not in each case be given, for instance let

$$P(x,y) = \frac{1}{(1/x) - (1/y)}$$

then $\mathbb{Q}_\infty \models P(0,0) = 0$. Indeed $\lim_{x\downarrow 0}(\lim_{y\downarrow 0} P(x,y)) = 0$ but consider the following path through this two dimensional limit: $\lim_{x\downarrow 0} P(x, x\cdot(1+x)) = 1$. It follows that the limit $\lim_{x\downarrow 0, y\downarrow 0} P(x,y)$ is not well-defined. Next consider

$$Q(x,y) = \frac{1}{(1/x)\cdot(1/(1/y))}$$

We find $\mathbb{Q}_\infty \models Q(0,0) = 0$ but $\lim_{x\downarrow 0}(\lim_{y\downarrow 0} Q(x,y)) = \infty$, whence the 2-dimensional limit $\lim_{x\downarrow 0, y\downarrow 0} Q(x,y)$ does not exist. One may conclude that justification of outcomes of evaluation in $\mathbb{Q}_\infty$ on the basis of asymptotic considerations is problematic. This observation motivates the design of wheels. In addition we notice both UFAR and CFAR fail in $\mathbb{Q}_\infty$:

$$0 = 0 + 0 = \frac{1}{\infty} + \frac{1}{\infty} \neq \frac{1\cdot\infty + \infty\cdot 1}{\infty\cdot\infty} = \frac{\infty\cdot\infty}{\infty} = \infty\cdot\frac{1}{\infty} = \infty\cdot 0 = \infty$$

To the best of our knowledge $\mathbb{Q}_\infty$ has not been investigated, or even proposed, in the literature on division by zero. Nevertheless it is a very plausible structure for which we suggest the following name: the uncommon wheel of rationals. The idea is that, in line with [25], an arithmetical datatype with a single non-finite value for the inverse of zero, i.e. a non-rational element e so that $e = 0^{-1}$, and $e^{-1} = 0$ is referred to as a wheel. Now the common wheels (simply referred to as wheels), as discussed below, are equipped with an absorbtive element $\bot$ while the uncommon wheels are not.

3.5 The wheel of rationals: combining ∞ and $\bot$

$\mathbb{Q}_\infty$ can be adapted in such a manner that problematic outcomes are avoided by incorporating $\bot$. This construction leads to the wheel of rationals (here denoted as $\mathbb{Q}_{\infty,\bot}$), as first specified in [25].

- $|\mathbb{Q}_{\infty,\bot}| = \mathbb{Q} \cup \{\infty, \bot\}$,
- $0^{-1} = \infty$,
- $\infty^{-1} = 0$,
- $\infty \cdot 0 = \bot$,
- $\infty + \infty = \bot$,
- $\infty \cdot \infty = \infty$,
- $-\infty = \infty$,
- for $x \in \mathbb{Q}, x \neq 0$: $\infty \cdot x = \infty$,

- for $x \in \mathbb{Q}$: $\infty + x = \infty$,
- $x + \bot = \bot + x = -\bot = x - \bot = \bot - x = x \cdot \bot = \bot \cdot x = \bot^{-1} = \bot/x = x/\bot = \bot$

These definitions suffice in combination with the requirement that $\mathbb{Q}_{\infty,\bot}$ is a premeadow which extends $\mathbb{Q}_\star$ and which extends the graphs of partial inverse and division to total functions.

It is worth mentioning that various familiar equations are invalid in the wheel of rationals, e.g. $x \cdot 0 = 0$, an the so-called quasi-cardinality rule $(x/y)+(z/y) = (x+z)/y$ (the name of this identity comes from [1]). We also note that UFAR and CFAR both fail in $\mathbb{Q}_{\infty,\bot}$:

$$\infty = 0 \cdot 0 + 0 \cdot 0 = \frac{0}{\infty} + \frac{0}{\infty} \neq \frac{0 \cdot \infty + \infty \cdot 0}{\infty \cdot \infty} = \frac{\bot + \bot}{\infty} = \frac{\bot}{\infty} = \bot$$

another example indicates that an occurrence of the constant 0 plays no role in this fact:

$$0 = 0 + 0 = \frac{1}{\infty} + \frac{1}{\infty} \neq \frac{1 \cdot \infty + \infty \cdot 1}{\infty \cdot \infty} = \frac{\infty + \infty}{\infty} = \frac{\bot}{\infty} = \bot$$

3.5.1 Semantic justification

The wheel of rationals seems to be justified in the following sense: it produces proper rational outcomes only in cases where limits are taken in arbitrary order and from an arbitrary direction, but we have no proof this somewhat informal intuition.

The metatheory of wheels is attractive and has been developed to a significant extent in [14, 15]. However, we are unaware of the existence of applications of wheels.

3.6 Transrationals: signed infinities combined with Φ

In some cases the wheel of rationals fails to provide a proper value (that is, a value in $\mathbb{Q}$) while it would be justified to do so. For instance the following identity can be justified if one thinks of ∞ as a positive infinite value which satisfies $\infty + \infty = \infty$:

$$\frac{1}{(1/0) + (1/0)} = 0$$

On the other hand there is less justification for having

$$\frac{1}{(1/0) + ((-1)/0)} = 0$$

By distinguishing positive and negative infinity both cases can be separated. If $1/0$ is positive infinity then $(-1)/0$ can be considered negative infinity, and while $\infty + \infty = \infty$ is plausible there is no plausible value for $\infty + (-\infty)$ except an error value. These considerations constitute the recent introduction of transrationals.

In the tradition of transrationals $\perp$ is denoted Φ, and so we may have both constants and require that $\perp = \Phi$.

We will denote the arithmetical datatype of transrationals with $\mathbb{Q}_{\pm\infty,\Phi}$. The domain includes on top of the known rationals three additional elements: $|\mathbb{Q}_{\pm\infty,\Phi}| = \mathbb{Q} \cup \{\infty, -\infty, \Phi\}$. We assume that the rational numbers are given as (n,m) with $n \in \mathbb{Z}$, $m \in \mathbb{N}$, $m > 0$ and $\gcd(m,n) = 1$. Thus 0 is represented by $(0,1)$ and 1 is represented by $(1,1)$. $S(n,m) = (n \backslash \gcd(n,m), m \backslash \gcd(n,m))$ denotes simplification of the pair (n,m) with $\_\backslash\_$ denoting integer division. The operations of $\mathbb{Q}_{\pm\infty,\Phi}$ are determined by the following 43 rewrite rules. In these rules the pairs (n,m) are considered auxiliary constants.

- $0 = (0,1), 1 = (1,1), \perp = \Phi$,
- $\Phi + x = \Phi$, $x + \Phi = \Phi$, $-\Phi = \Phi$, $\Phi \cdot x = \Phi$, $x \cdot \Phi = \Phi$, $\Phi^{-1} = \Phi$,
- $\infty \cdot \infty = \infty, \infty \cdot -\infty = -\infty, -\infty \cdot \infty = -\infty, -\infty \cdot -\infty = \infty$,
- $\infty \cdot (0,1) = \Phi, (0,1) \cdot \infty = \Phi, -\infty \cdot (0,1) = \Phi, (0,1) \cdot -\infty = \Phi$,
- $n > 0 \to \infty \cdot (n,m) = \infty$, $n > 0 \to (n,m) \cdot \infty = \infty$,
- $n > 0 \to -\infty \cdot (n,m) = -\infty$, $n > 0 \to (n,m) \cdot -\infty = -\infty$,
- $n < 0 \to \infty \cdot (n,m) = -\infty$, $n > 0 \to (n,m) \cdot \infty = \infty$,
- $n < 0 \to -\infty \cdot (n,m) = \infty$, $n > 0 \to (n,m) \cdot -\infty = -\infty$,
- $\infty + (-\infty) = \Phi, (-\infty) + \infty = \Phi$,
- $\infty + \infty = \infty, (-\infty) + (-\infty) = -\infty$,
- $\infty + (n,m) = \infty$, $(n,m) + \infty = \infty$,
- $-\infty + (n,m) = -\infty$, $(n,m) + -\infty = -\infty$,
- $(0,1)^{-1} = \infty$,
- $\infty^{-1} = (0,1), (-\infty)^{-1} = (0,1)$,
- $-(n,m) = (-n,m)$,
- $(n,m) + (k,l) = S(n \cdot k + m \cdot l, m \cdot l)$,
- $(n,m) \cdot (k,l) = S(n \cdot k, m \cdot l)$,
- $n > 0 \to (n,m)^{-1} = (m,n)$, $n < 0 \to (n,m)^{-1} = (-m,-n)$,
- $x - y = x + (-y)$, $x/y = x \cdot y^{-1}$.

Here it is understood that say $(n,m) \cdot (k,l) = S(n \cdot k, m \cdot l)$ represents a single step for each pair (n,m) with $m \neq 0$. It is easy to see that $\mathbb{Q}_{\pm\infty,\Phi}$ is a premeadow. The rewrite system is confluent and terminating.

Proposition 3.1. *For each closed expression t not involving Φ and $\bot$: if $\mathbb{Q}_{\pm\infty,\Phi} \models t = \Phi$ then $\mathbb{Q}_{\text{inf},\bot} \models t = \bot$.*

Proof. Straightforward with induction on the length of a reduction path to normal form in the rewrite system for transrationals. □

3.6.1 Semantic justification

Providing a semantic justification for $\mathbb{Q}_{\pm\infty,\Phi}$ is not entirely straightforward. For instance consider the expression $Q(x,y) = (x^{-1} + (((-1)/y)^{-1})^{-1})^{-1}$. Evaluation of $Q(0,0)$ in the premeadow of transrationals yields:

$$Q(0,0) = \frac{1}{0^{-1} + (((-1)/0)^{-1})^{-1}} = \frac{1}{\infty + ((-\infty)^{-1})^{-1}} = \frac{1}{\infty + \infty} = \frac{1}{\infty} = 0$$

On the other hand $\lim_{x\downarrow 0, y\downarrow 0} R(x,y) = \lim_{x\downarrow 0, y\downarrow 0} \frac{1}{x^{-1} - y^{-1}}$ and the latter limit does not exist. It appears that for explaining the results of evaluation in the arithmetical datatype of transrationals it is important one must have an inside out evaluation strategy in mind. Moreover a preference for innermost evaluation takes priority over considerations in terms of limits of related rational functions. Innermost evaluation requires that an expression is only evaluated after each of its subexpressions has been evaluated.

Each of the identities listed in the description can be given an explanation in terms of limits and then by adopting the principle of innermost evaluation one obtains a justification for each derivable identity between closed terms in $\mathbb{Q}_{\pm\infty,\Phi}$.

3.7 Beyond transrationals? Incorporating a negative zero

Consider the fraction

$$P(x,y) = \frac{1}{(\frac{1}{x} + \frac{-1}{(\frac{1}{((-1)/y)})})}$$

Evaluation of the closed fraction $P(0,0)$ in the premeadow of transrationals yields:

$$P(0,0) = \frac{1}{\frac{1}{0} + \frac{-1}{(\frac{1}{((-1)/0)})}} = \frac{1}{\infty + \frac{-1}{(\frac{1}{-\infty})}} = \frac{1}{\infty + \frac{-1}{0}} = \frac{1}{\infty + (-\infty)} = \frac{1}{\Phi} = \Phi$$

Now $\lim_{x\downarrow 0, y\downarrow 0} P(x,y) = \lim_{x\uparrow 0, y\uparrow 0} P(x,y) = 0$ from which it may be inferred that there is ample asymptotic justification for having $P(0,0) = 0$, and that a reasonable case can be made to look for a further refinement or adaptation of transrational arithmetic in which $P(0,0)$ evaluates to 0 rather than to Φ. We notice that $\mathbb{Q}_0 \models P(0,0) = 0$ but one may claim with ample justification that $\mathbb{Q}_0$ allows too many identities, so that contemplating $\mathbb{Q}_0$ does not provide a useful solution to this question, and neither does working in $Q_{\infty,\bot}$ where $P(0,0)$ evaluates to $\bot$, because $Q_{\infty,\bot}$ fails to differentiate between the two expressions discussed in the beginning of Paragraph 3.6.

There is room for modification of the transrational arithmetic as defined in [2] in order to take alternative requirements regarding the evaluation of closed expressions into account.

Problem 3.1. *Does there exist an arithmetical datatype which satisfies $P(0,0) = 0$ and in which all or most expressions which evaluate in transrational arithmetic to a finite non-error result evaluate to the same result, while still sufficiently many expressions evaluate to an error (i.e. to Φ).*

At first sight the introduction of a negative zero, as promoted for instance in the IEEE-754 floating point standard, may easily lead to a solution to this question, but the details involved are not straightforward. We refer to the filter calculus of [3] as work which is relatively close to this topic.

3.8 Transrational arithmetic with a second zero

As an experiment we introduce the additional constant $\circleddash$ which represents negative zero, or rather the inverse of negative infinity, and which we will call "second zero", in order to avoid suggesting the plausibility of certain equations, such as for instance $-\circleddash = 0$ or $(-\circleddash)^{-1} = \infty$. Because the axioms below imply that $\circleddash = (-(0^{-1})^{-1}$ this constant logically serves as an abbreviation. Its role in term rewriting is different and more important as it may serve as a normal form in circumstances where the expression $(-(0^{-1})^{-1}$ of which it serves as an abbreviation is not considered a normal form. The rewrite system is adapted by including in addition the following 18 rewrite rules:

- $\infty \cdot \circleddash = \Phi, \circleddash \cdot \infty = \Phi, -\infty \cdot \circleddash = \Phi, \circleddash \cdot -\infty = \Phi$,
- $\infty + \circleddash = \infty$, $\circleddash + \infty = \infty$,
- $(-\infty) + \circleddash = -\infty$, $\circleddash + (-\infty) = -\infty$,
- $\circleddash^{-1} = -\infty$,
- $\infty^{-1} = (0,1), (-\infty)^{-1} = \circleddash$,
- $-\circleddash = \circleddash$,
- $\circleddash + (n,m) = \circleddash$, $(n,m) + \circleddash = \circleddash$,
- $\circleddash + \circleddash = \circleddash$,
- $\circleddash \cdot \circleddash = \circleddash$,
- $\circleddash \cdot (n,m) = \circleddash$, $(n,m) \cdot \circleddash = \circleddash$,

Together these rules will be called $\mathsf{TM}_{\circleddash}$. The rewrite system is terminating and confluent. The arithmetic datatype thus obtained, will be denoted with $\mathbb{Q}_{\pm\infty,\circleddash,\Phi}$ is a premeadow. $\mathbb{Q}_{\pm\infty,\circleddash,\Phi}$ satisfies $P(0,0) = 0$. However, in contrast with transrational arithmetic $\mathbb{Q}_{\pm\infty,\circleddash,\Phi} \models Q(0,0) = \Phi$ (with $Q(x,y)$ as in Paragraph 3.6.1). Compared with $\mathbb{Q}_{\pm\infty,\Phi}$, $\mathbb{Q}_{\pm\infty,\circleddash,\Phi}$ is a less natural arithmetical datatype because of the asymmetry between ∞^{-1} and $(-\infty)^{-1}$. For instance in $\mathbb{Q}_{\pm\infty,\ominus,\Phi}$, $1+\infty^{-1}=1$ and $1+(-\infty)^{-1}=(-\infty)^{-1}$, with the latter equation having no asymptotic justification at all. $\mathbb{Q}_{\pm\infty,\ominus,\Phi}$ qualifies as a solution of Problem 3.1 but it is probably too asymmetric to be useful.

Finding adaptations of transrational arithmetic in which there is a meaningful role for a second zero, which may behave more like a negative zero than $\ominus$, is at the time of writing this survey an open topic for further research, which in all likelihood allows for a plurality of different designs.

The arithmetical datatype $\mathbb{Q}_{\pm\infty,\ominus,\Phi}$, which we will call transrationals (transrational arithmetic) with a second zero is merely one option indicating such possibilities, and no claim is made that $\mathbb{Q}_{\pm\infty,\ominus,\Phi}$ will constitute a prominent outcome of such investigations.

4 Concluding remarks

The theme of division by zero is introduced and the conventional approach to this matter is sketched. The central role of the appreciation of a contrast between syntax and semantics is emphasised and an attempt is made to formulate in what sense elaborating on the theme of division by zero is reasonably possible.

Using the notion of a premeadow, which generalises the specailization to the assiciative case of the notion of a transfield as proposed in [21], a brief survey is given of options for defining arithmetical datatypes providing a total extension of the division operator. The survey is restricted to characteristic zero and to datatypes which extend the rational numbers by one or finitely more additional elements.

Our survey of options is incomplete, first of all because the literature on division by zero is already hard to grasp and quite diverse, but more importantly because some possible avenues have yet been left unexplored, as is exemplified by the role of signed zeroes, which feature in practice more than in theory. An example is given of an arithmetical datatype which contains a negative zero.

References

[1] H. Athen and H. Griesel (Hrsg.) *Mathematik Heute 6. (In German.)* Schroedel Verlag Hannover & Schöning Verlag Paderborn, (1978).

[2] J.A. Anderson, N. Völker, and A. A. Adams. Perspecx Machine VIII, axioms of transreal arithmetic. Vision Geometry XV, eds. J. Latecki, D. M. Mount and A. Y. Wu, 649902; https://doi.org/10.1117/12.698153 (2007).

[3] M. Beeson and F. Wiedijk. The meaning of infinity in calculus and computer algebra systems. *Journal of Symbolic Computation.* 39, 523–538 (2005).

[4] J.A. Bergstra and I. Bethke. Subvarieties of the variety of meadows. *Scientific Annals of Computer Science.* vol 27 (1) 1–18, (2017).

[5] J. A. Bergstra, I. Bethke, and A. Ponse. Equations for formally real meadows. *Journal of Applied Logic*, 13 (2): 1–23, (2015).

[6] J.A. Bergstra, Y. Hirshfeld, and J.V. Tucker. Meadows and the equational specification of division. *Theoretical Computer Science*, 410 (12), 1261–1271 (2009).

[7] J.A. Bergstra and C.A. Middelburg. Division by zero in non-involutive meadows. *Journal of Applied Logic*, 13 (1): 1–12 (2016).

[8] J.A. Bergstra and C.A. Middelburg. Transformation of fractions into simple fractions in divisive meadows. *Journal of Applied Logic*, 16: 92–110 (2015). (Also https://arxiv.org/abs/1510.06233.)

[9] J.A. Bergstra and A. Ponse. Fracpairs and fractions over a reduced commutative ring. *Indigationes Mathematicae* 27, 727-748, (2016). http://dx.doi.org/10.1016/j.indag.2016.01.007. (Also https://arxiv.org/abs/1411.4410).

[10] J.A. Bergstra and A. Ponse. Division by zero in common meadows. *In R. de Nicola and R. Hennicker (editors), Software, Services, and Systems (Wirsing Festschrift),* LNCS 8950, pages 46-61, Springer, 2015. Also available at arXiv:1406.6878v2[math.RA], (2015).

[11] J.A. Bergstra and J.V. Tucker. The rational numbers as an abstract data type. *Journal of the ACM*, 54 (2), Article 7 (2007).

[12] I. Bethke and P.H. Rodenburg. The initial meadows. *Journal of Symbolic Logic,* 75 (3), 888-895, (2010).

[13] C.B. Boyer. An ancient reference to division by zero. *The American Mathematical Monthly,* 50 (8), 487-491, (1943).

[14] J. Carlström. Wheels–on division by zero. *Math. Structures in Computer Science*, 14 (1) pp. 143–184, (2004).

[15] J. Carlström. Partiality and choice, foundational contributions. *PhD. Thesis, Stockholm University*, http://www.diva-portal.org/smash/get/diva2:194366/FULLTEXT01.pdf, (2005).

[16] J. Czajko. On Cantorian spacetime over number systems with division by zero. *Chaos, Solitons and Fractals*, 21 pp. 261–271, (2004).

[17] H. D. Ehrich, M. Wolf, and J. Loeckx. Specification of Abstract Data Types. *Vieweg + Teubner*, ISBN-10:3519021153, (1997)

[18] Institute of Reproducing Kernels (Anonymous). Announcement 412: The 4th birthday of the division by zero $z/0 = 0$ (2018.2.2). *Institute of reproducing kernels, Kawauchi-cho 5-1648-16, Kiryu 376-0041, Japan*, (2018).

[19] Y. Komori. Free algebras over all fields and pseudo-fields. Report 10, pp. 9-15, Faculty of Science, Shizuoka University (1975).

[20] H. Michiwaki, S. Saitoh, and N. Yamada. Reality of the division by zero $z/0 = 0$. *International Journal of Applied Physics and mathematics*, doi:10.17706/ijapm.2016.6.1.1-8 (2016).

[21] T.S. dos Reis, W. Gomide, and J.A.D.W. Anderson. Construction of the transreal numbers and algebraic transfields. *IAENG International Journal of Applied Mathematics*, 46 (1), 11–23, (2016). (http://www.iaeng.org/IJAM/issues_v46/issue_1/IJAM_46_1_03.pdf)

[22] H. Ono. Equational theories and universal theories of fields. *Journal of the Mathematical Society of Japan*, 35(2), 289-306 (1983).

[23] Saburou Saitoh. Who did derive First the Division by Zero $1/0$ and the Division by Zero Calculus $\tan(\pi/2) = 0$, $\log 0 = 0$ as the outputs of a computer. http://vixra.org/pdf/1903.0184v1.pdf (2019).

[24] B. Santangelo. The axiomatic construction of a new algebraic structure in order to extend a field and define division by zero. https://arxiv.org/pdf/1611.06838.pdf (2016).

[25] A. Setzer. Wheels (draft). http://www.cs.swan.ac.uk/ csetzer/articles/wheel.pdf, (1997).

[26] Okoh Ufuoma. On the operation of division by zero in Bhaskara's framework: criticisms, modifications, and justifications. *Asian Research Journal of Mathematicals*, 6 (2), 1–20, Article no. AJROM.36240 (2017). DOI: 10.9734/ARJOM/2017/36240.