Transreal Integral

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Abstract

Transreal integrals seek to extend the usual definitions of integrals from real to transreal numbers. An earlier definiton of the transreal integral was only partially successful. It obtained many real integrals but could not obtain improper integrals of some real functions. The present integral overcomes these limitations. It is defined on the entire set of transreal numbers and integrates all functions which are properly or improperly integrable, in the usual sense, on real numbers.

1 Introduction

Real numbers are used in everyday life to count and measure finite things. In some areas of mathematics and science, the real numbers are extended with a positive and a negative infinity that make sense of unboundedly large magnitudes. The transreal numbers take this further: they contain all of the real numbers, the infinities of the extended real numbers, and a new non-finite number, namely nullity. Thus the transreal numbers extend counting and measuring from the finite world of everyday experience to a hypothetical and, perhaps, actual world that contains everyday finite quantities and both infinite and nullity values at singularities.

Now that we have access to transreal numbers, it seems natural to ask what operations on real numbers can be extended to transreal numbers. In particular, what parts of real calculus – which is widely used in science and engineering to describe the workings of the physical world and to design machines and structures – can be extended to deal with infinite and nullity singularities.

In the papers [1] [3] [4] [5] we have been extending the real calculus to a transreal calculus. We have defined topology, limits, continuity, derivative and integral on transreal numbers. We aimed to extend the usual definitions on real numbers so that when any of transreal topology, limit, continuity or derivative is applied to real numbers the result is the same as when using the usual, real, definition. All real numbers are transreal numbers, so the transreal derivative, for example, can be applied to real numbers. When we take the transreal derivative of an arbitrary real function of a real variable, we get exactly the usual, real, derivative of that function. The same occurs for topology, limits and continuity. Similarly transreal topology and limits give the usual answers when applied to extended real numbers that admit infinities. However, the same does not occur for all improperly integrable functions. According to the earlier definition of the integral on transreal numbers, some improperly integrable (in the usual sense) functions are not integrable in the transreal sense. For example, the function $x \mapsto \frac{\sin x}{x}$ is improperly integrable in $[0, \infty]$ but is not integrable in the earlier transreal sense. In [2] we proposed an integral on extended real numbers. All properly or improperly integrable functions are integrable in the sense proposed in [2]. Here we extend that integral to the transreal numbers. In this way, the present paper introduces an integral, defined on transreal numbers, that contains the real integral, as promised in the first paragraph of the Section V of [5]. All properly or improperly integrable functions, in the usual real and extended real senses, are integrable in the transreal sense, introduced here.

2 Preliminaries

The set of transreal numbers, denoted $\mathbb{R}^T$, is formed by the real numbers and the three new elements minus infinity, infinity and nullity, which are denoted, respectively, by $-\infty$, ∞ and Φ. Therefore $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$. Division by zero is allowed in the set of transreal numbers. Specifically $-1/0 = -\infty$, $1/0 = \infty$ and $0/0 = \Phi$. The arithmetic and order relation defined on $\mathbb{R}^T$ are such that for each $x, y \in \mathbb{R}^T$ it follows that:

i) If $x \in \mathbb{R}$ then $-\infty < x < \infty$.

ii) The following does not hold $x < \Phi$ or $\Phi < x$.

iii) $-(\infty) = -\infty$, $\quad -(-\infty) = \infty \quad$ and $\quad -\Phi = \Phi$.

iv) $\infty^{-1} = 0$, $\quad (-\infty)^{-1} = 0$, $\quad \Phi^{-1} = \Phi \quad$ and $\quad 0^{-1} = \infty$.

v) $\infty + x = \begin{cases} \Phi, & \text{if } x \in \{-\infty, \Phi\} \\ \infty, & \text{otherwise} \end{cases}$, $\quad -\infty + x = -(\infty - x) \quad$ and $\Phi + x = \Phi$.

vi) $\infty \times x = \begin{cases} \Phi, & \text{if } x \in \{0, \Phi\} \\ -\infty, & \text{if } x < 0 \\ \infty, & \text{if } x > 0 \end{cases}$, $\quad -\infty \times x = -(\infty \times x) \quad$ and $\Phi \times x = \Phi$.

vii) $x - y = x + (-y)$.

viii) $x \div y = x \times y^{-1}$.

The consistency of transreal arithmetic is proved in [6].

For all $x, y \in \mathbb{R}^T$, we write $x \not< y$ if and only if $x < y$ does not hold and we write $x \not> y$ if and only if $x > y$ does not hold. Notice that $\not<$ is not equivalent to $\geq$. For example $\Phi \not< 0$ but $\Phi \geq 0$ does not hold. However, for all $x, y \in \mathbb{R}^T \setminus \{\Phi\}$, it follows that $x \not< y$ if and only if $x \geq y$ and $x \not> y$ if and only if $x \leq y$.

In Definition 14 in [3] and in Definition 21 in [5] we defined the supremum and infimum on transreal numbers but there is a mistake there. Those definitions must be replaced by Definition 1 below.

Definition 1. Let $A \subset \mathbb{R}^T$ be an arbitrary non-empty set. We say that $u \in \mathbb{R}^T$ is the *supremum* of A and we write $u = \sup A$ if and only if one of the following conditions occurs:

i) $A = \{\Phi\}$ and $u = \Phi$ or

ii) $x \leq u$ for all $x \in A \setminus \{\Phi\}$ and for each $y \in [-\infty, u)$ there is $x \in A$ such that $y < x$.

And we say that $v \in \mathbb{R}^T$ is the *infimum* of A and we write $v = \inf A$ if and only if one of the following conditions occurs:

iii) $A = \{\Phi\}$ and $v = \Phi$ or

iv) $v \leq x$ for all $x \in A \setminus \{\Phi\}$ and for each $y \in (v, \infty]$ there is $x \in A$ such that $x < y$.

Notice that for all $A \subset \mathbb{R}^T$ if $A \setminus \{\Phi\} \neq \varnothing$ then $\sup A = \sup A \setminus \{\Phi\}$ and $\inf A = \inf A \setminus \{\Phi\}$.

Let $a, b \in \mathbb{R}^T$. We define

a) $(a, b) := \{x \in \mathbb{R}^T;\ a < x < b\}$

b) $(a, b] := (a, b) \cup \{b\}$,

c) $[a, b) := \{a\} \cup (a, b)$ and

d) $[a, b] := \{a\} \cup (a, b) \cup \{b\}$.

Notice that for all $a \in \mathbb{R}^T$ we have that $[a, \Phi] = \{a, \Phi\}$ and $[\Phi, a] = \{a, \Phi\}$. If we had defined $[a, b] = \{x \in \mathbb{R}^T;\ a \leq x \leq b\}$, we would have $[a, \Phi] = \varnothing$ for all $a \in \mathbb{R}^T$.

The transreal numbers are a topological space where the open subsets are arbitrary unions of finitely many intersections of the following four kinds of intervals:

i) (a, b) where $a, b \in \mathbb{R}$,

ii) $[-\infty, b)$ where $b \in \mathbb{R}$,

iii) $(a, \infty]$ where $a \in \mathbb{R}$ and

iv) $\{\Phi\}$.

The topology of $\mathbb{R}^T$ contains the topology of $\mathbb{R}$, that is, when it is restricted to subsets of $\mathbb{R}$, it coincides with the topology of $\mathbb{R}$.

The definition for the convergence of a sequence is the usual in a topological space. That is a sequence, $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$, converges to $x \in \mathbb{R}^T$ if and only if for each neighbourhood, $V \subset \mathbb{R}^T$ of x, there is $n_V \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_V$. Notice that if $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $L \in \mathbb{R}$ then $\lim_{n\to\infty} x_n = L$ in $\mathbb{R}^T$ if and only if $\lim_{n\to\infty} x_n = L$ in the usual sense in $\mathbb{R}$. Furthermore, $(x_n)_{n\in\mathbb{N}}$ diverges, in the usual sense, to negative infinity if and only if $\lim_{n\to\infty} x_n = -\infty$ in $\mathbb{R}^T$. Similarly $(x_n)_{n\in\mathbb{N}}$ diverges, in the usual sense, to infinity if and only if $\lim_{n\to\infty} x_n = \infty$ in $\mathbb{R}^T$. Notice also that $\lim_{n\to\infty} x_n = \Phi$ if and only if there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$.

Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$. For each $n \in \mathbb{N}$, we define $s_n := \sum_{i=1}^{n} x_i$. The sequence $(s_n)_{n\in\mathbb{N}}$ is called a series and is denoted by $\sum x_n$. Each s_n is called a partial sum of $\sum x_n$ and x_n is called the n-th term of $\sum x_n$. We say that $\sum x_n$ converges or is convergent if and only if there is the $\lim_{n\to\infty} s_n$. Otherwise, $\sum x_n$ diverges or is divergent. When $\sum x_n$ is convergent we denote $\sum_{n=1}^{\infty} x_n := \lim_{n\to\infty} s_n$.

Let $(x_n)_{n\in\mathbb{Z}} \subset \mathbb{R}^T$. For each $n \in \mathbb{N}$ we denote $r_n := \sum_{i=1}^{n} x_{-i}$ and $t_n := \sum_{i=1}^{n} x_{i-1}$. The pair $(\sum x_{-n}\ ,\ \sum x_{n-1})$ is called a bilateral series. We say that $(\sum x_{-n}\ ,\ \sum x_{n-1})$ converges if and only if $\sum x_{-n}$ and $\sum x_{n-1}$ are both convergent. In this way we denote $\sum_{n=-\infty}^{\infty} x_n := \sum_{n=1}^{\infty} x_{-n} + \sum_{n=1}^{\infty} x_{n-1}$. We also denote $\sum_{n=-\infty}^{-1} x_n := \sum_{n=1}^{\infty} x_{-n}$ and $\sum_{n=0}^{\infty} x_n := \sum_{n=1}^{\infty} x_{n-1}$. Frequently we abuse notation and denote a bilateral series $(\sum x_{-n}\ ,\ \sum x_{n-1})$ by $\sum_{n=-\infty}^{\infty} x_n$.

3 The Transreal Integral

Definition 2. Let $a, b \in \mathbb{R}^T$ where $a \not> b$ and $f : [a, b] \to \mathbb{R}^T$.

a) A sequence $(x_n)_{n\in\mathbb{Z}} \subset \mathbb{R}^T$ is called an *transpartition* of $[a, b]$ if and only if it satisfies all of the three following conditions:
   i) $(x_n)_{n\in\mathbb{Z}} \subset [a, b]$,
   ii) $x_i \not> x_j$ whenever $i < j$ and
   iii) $\lim_{n\to-\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$.

For each $(x_n)_{n\in\mathbb{Z}}$ transpartition of $[a, b]$ denote, for each $i \in \mathbb{Z}$:

b) $\Delta x_i := x_i - x_{i-1}$,

c) $m_i := \inf\big\{f(x);\ x \in [x_{i-1}, x_i]\big\}$,

d) $M_i := \sup\big\{f(x);\ x \in [x_{i-1}, x_i]\big\}$.

e) Denote the set of all transpartitions $(x_n)_{n\in\mathbb{Z}}$ of $[a, b]$ such that $\sum_{i=-\infty}^{\infty} m_i \Delta x_i$ and $\sum_{i=-\infty}^{\infty} M_i \Delta x_i$ are both convergent in $\mathbb{R}^T$ as $\mathcal{P}_{\mathbb{R}^T}\big(f; [a, b]\big)$.

For each $P = (x_n)_{n\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}^T}\big(f; [a, b]\big)$, denote

f) $L\big(f; P\big) := \sum_{i=-\infty}^{\infty} m_i \Delta x_i$ and

g) $U\big(f; P\big) := \sum_{i=-\infty}^{\infty} M_i \Delta x_i$.

Definition 3. Let $a, b \in \mathbb{R}^T$ where $a \not> b$ and $f : [a, b] \to \mathbb{R}^T$. The function f is said to be *integrable* in $[a, b]$ if and only if

$$\sup\big\{L(f; P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f; [a, b]\big)\big\} = \inf\big\{U(f; P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f; [a, b]\big)\big\}.$$

And in this case the *integral* of f in $[a, b]$ is defined by

$$\int_a^b f := \sup\big\{L(f; P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f; [a, b]\big)\big\}.$$

Henceforth the integral defined above is called simply an *integral* and is denoted by $\int_a^b f$. The *Riemann integral* is denoted by $\underset{\mathcal{R}}{\int_a^b} f$.

The integrable functions, in the sense of Definition 3, make a superset of the Riemann, proper or improper, integrable functions. That is, this paper's definition integrates every function which Riemann integrates properly or improperly.

Theorem 4. Let $a, b \in \mathbb{R}^T \setminus \{\Phi\}$ where $a < b$ and $f : [a, b] \to \mathbb{R}^T \setminus \{\Phi\}$ such that $f\big((a, b)\big) \subset \mathbb{R}$. If f is Riemann integrable, either as a proper integral or as an improper integral, then f is integrable (in the sense of Definition 3) and

$$\int_a^b f = \underset{\mathcal{R}}{\int_a^b} f.$$

Proof. Let $a, b \in \mathbb{R}^T \setminus \{\Phi\}$ where $a < b$ and $f : [a, b] \to \mathbb{R}^T \setminus \{\Phi\}$ such that $f\big((a, b)\big) \subset \mathbb{R}$ and f is Riemann integrable, either as a proper integral or as an improper integral.

Since $a \neq b$ and $\Phi \notin [a, b]$ we can suppose, without loss of generality, that $x_n < x_{n+1}$ for all $n \in \mathbb{N}$ whatever $P = (x_n)_{n\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}^T}\big(f; [a, b]\big)$. In this way, the result follows from Theorem 2.1 of [2]. $\square$

The Theorem 4 shows that the integral defined in the transreal domain agrees with the usual integral when applied on real numbers.

Next, we show several cases where the integral result is nullity.

Theorem 5. Let $a, b \in \mathbb{R}^T$ where $a \not> b$ and $f : [a, b] \to \mathbb{R}^T$. If one of the following conditions occurs

I) $a = \Phi$,

II) $b = \Phi$,

III) $a = b$ and $a \notin \mathbb{R}$,

IV) $a = b$ and $f(a) \notin \mathbb{R}$

V) there is $c \in (a, b)$ such that $f(x) = \Phi$ for all $x \in [a, c]$,

VI) there is $c \in (a, b)$ such that $f(x) = \Phi$ for all $x \in [c, b]$,

VII) $f(x) = \Phi$ for all $x \in (a, b)$ and $|f(a)f(b)| = \infty$,

then f is integrable and $\int_a^b f = \Phi$.

Proof. Let $a, b \in \mathbb{R}^T$ where $a \not> b$ and $f : [a, b] \to \mathbb{R}^T$.

I) Suppose $a = \Phi$. For all $P = (x_n)_{n\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}^T}\big(f; [a, b]\big)$ it follows that $\lim_{n\to-\infty} x_n = a = \Phi$ whence there is $n_0 \in \mathbb{Z}$ such that $x_n = \Phi$ for all $n \leq n_0$.

Thus

$$\begin{aligned}
L\big(f;P\big) &= \sum_{i=-\infty}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_0} m_i \Delta x_i + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_0} m_i (x_i - x_{i-1}) + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_0} m_i (\Phi - \Phi) + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_0} m_i \Phi + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_0} \Phi + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \Phi + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \Phi
\end{aligned}$$

for all $P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$. Thereby $\big\{L(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \{\Phi\}$ whence

$$\sup\big\{L(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

In an analogous way we can see that

$$\inf\big\{U(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

Therefore f is integrable and $\int_a^b f = \Phi$.

II) The proof is analogous to item I.

III) Suppose $a, b \notin \mathbb{R}$ and $a = b$. If $a = \Phi$ then the result is already proved above. If $a = -\infty$ then $b = -\infty$. For all $P = (x_n)_{n\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$ it follows that $(x_n)_{n\in\mathbb{Z}} \subset [a,b] = [-\infty,-\infty] = \{-\infty\}$ whence $x_n = -\infty$ for all $n \in \mathbb{Z}$.

Thus

$$\begin{aligned} L\big(f;P\big) &= \sum_{i=-\infty}^{\infty} m_i \Delta x_i \\ &= \sum_{i=-\infty}^{\infty} m_i (x_i - x_{i-1}) \\ &= \sum_{i=-\infty}^{\infty} m_i (-\infty - (-\infty)) \\ &= \sum_{i=-\infty}^{\infty} m_i (-\infty + \infty) \\ &= \sum_{i=-\infty}^{\infty} m_i \Phi \\ &= \sum_{i=-\infty}^{\infty} \Phi \\ &= \Phi \end{aligned}$$

for all $P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$. Thereby

$$\sup\big\{L(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

In an analogous way we can see that

$$\inf\big\{U(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

Therefore f is integrable and $\int_a^b f = \Phi$. If $a = \infty$ then, in an analogous way, we can see that f is integrable and $\int_a^b f = \Phi$.

IV) Suppose $a = b$ and $f(a) \notin \mathbb{R}$. For all $P = (x_n)_{n\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$, since $a = b$, it follows that $(x_n)_{n\in\mathbb{Z}} \subset [a,a] = \{a\}$, that is, $x_n = a$ for all $n \in \mathbb{Z}$ whence $\Delta x_n = a - a \in \{0, \Phi\}$ and since $f(a) \notin \mathbb{R}$, it follows that $m_n = f(a) \notin \mathbb{R}$ for all $n \in \mathbb{Z}$ whence $m_n \Delta x_n = \Phi$ for all $n \in \mathbb{Z}$. Thus $L\big(f;P\big) = \Phi$ for all $P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$. Thereby

$$\sup\big\{L(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

In an analogous way we can see that

$$\inf\big\{U(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

Therefore f is integrable and $\int_a^b f = \Phi$.

V) Suppose there is $c \in (a,b)$ such that $f(x) = \Phi$ for all $x \in [a,c]$. Since $c \in (a,b)$ we have that $a < c < b$ whence $c \in \mathbb{R}$ and $a \neq \Phi$. For all $P = (x_n)_{n\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$ it follows that $(x_n)_{n\in\mathbb{Z}} \subset [a,b]$ and $\lim_{n\to-\infty} x_n = a$ whence there is $n_0 \in \mathbb{Z}$ such that $x_n \in [a,c)$ for all $n \leq n_0$.

Thus

$$\begin{aligned}
L\big(f;P\big) &= \sum_{i=-\infty}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_0} m_i \Delta x_i + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_0} \Phi \Delta x_i + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\
&= \Phi
\end{aligned}$$

for all $P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$. Thereby

$$\sup\big\{L(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

In an analogous way we can see that

$$\inf\big\{U(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

Therefore f is integrable and $\int_a^b f = \Phi$.

VI) The proof is analogous to item V.

VII) Suppose $f(x) = \Phi$ for all $x \in (a,b)$ and $|f(a)f(b)| = \infty$. If $a = \Phi$ or $b = \Phi$ or $a = b$ the the results follows from item I or II or IV, respectively. Otherwise, since $|f(a)f(b)| = \infty$ it follows that $|f(a)||f(b)| = \infty$ whence either $f(a) = \infty$ or $f(a) = -\infty$ or $f(b) = \infty$ or $f(b) = -\infty$.

If $f(a) = \infty$ then, for all $P = (x_n)_{n\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$, either $(x_n)_{n\in\mathbb{Z}} \subset (a,b]$ or there is $n_0 \in \mathbb{Z}$ such that $x_{n_0} = a$. If $(x_n)_{n\in\mathbb{Z}} \subset (a,b]$ then there is $n_1 \in \mathbb{Z}$ such that $x_{n_1} \in (a,b)$ whence $f(x) = \Phi$ for all $x \in [x_{n_1-1}, x_{n_1}]$. Hence $m_{n_1} = \Phi$ whence

$$\begin{aligned}
L\big(f;P\big) &= \sum_{i=-\infty}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_1-1} m_i \Delta x_i + m_{n_1}\Delta x_{n_1} + \sum_{i=n_1+1}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_1-1} m_i \Delta x_i + \Phi \times \Delta x_{n_1} + \sum_{i=n_1+1}^{\infty} m_i \Delta x_i \\
&= \sum_{i=-\infty}^{n_1-1} m_i \Delta x_i + \Phi + \sum_{i=n_1+1}^{\infty} m_i \Delta x_i \\
&= \Phi.
\end{aligned}$$

If there is $n_0 \in \mathbb{Z}$ such that $x_{n_0} = a$ then $x_{n_0-1} = a$ whence $\Delta x_{n_0} = a - a \in \{0, \Phi\}$. Since $m_{n_0} = f(a) = \infty$, $m_{n_0}\Delta x_{n_0} = \Phi$.

$$\begin{aligned} L\big(f;P\big) &= \sum_{i=-\infty}^{\infty} m_i \Delta x_i \\ &= \sum_{i=-\infty}^{n_0-1} m_i \Delta x_i + m_{n_0}\Delta x_{n_0} + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\ &= \sum_{i=-\infty}^{n_0-1} m_i \Delta x_i + \Phi + \sum_{i=n_0+1}^{\infty} m_i \Delta x_i \\ &= \Phi. \end{aligned}$$

Thus $L\big(f;P\big) = \Phi$ for all $P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)$. Thereby

$$\sup\big\{L(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

In an analogous way we can see that

$$\inf\big\{U(f;P);\ P \in \mathcal{P}_{\mathbb{R}^T}\big(f;[a,b]\big)\big\} = \Phi.$$

Therefore f is integrable and $\int_a^b f = \Phi$.

In an analogous way we can see that f is integrable and $\int_a^b f = \Phi$ if $f(a) = -\infty$ or $f(b) = \infty$ or $f(b) = -\infty$.

□

4 Conclusion

An integral on transreal numbers was first defined in [5]. That integral is not the most general one because there are functions integrable in the usual sense which are not integrable in the sense of [5]. The present paper has taken the approach from [2] and defined an integral on transreal numbers which generalises the usual integral on real numbers. Every function integrable in the usual (Riemann) sense, properly or improperly, is integrable in the sense introduced here. In addition, several arrangements of transreal numbers make the integral results nullity.

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