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Review of Henry Thomas Colebrooke's 1817 Translation of Sanscrit Works on Division by Zero

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Abstract

We review the introduction of the number zero and of total arithmetical operations of addition, subtraction, multiplication, and division in various Sanscrit works from 628 CE to 1621 CE, translated into English in the book "ALGEBRA WITH ARITHMETIC AND MENSURATION, FROM THE SANSCRIT OF BRAHMEGUPTA AND BHASCARA" by Henry Thomas Colebrooke, 1817 CE. We note the existence of intermediate steps in the development of zero from a placeholder into a number. We introduce the concepts of a paraconsistent arithmetic that is uniquely determined but which contradicts its axioms, and of a paralogical arithmetic that is uniquely determined but which is non-logical. We find that Brahmegupta, 628 CE, described two total paraconsistent arithmetics of fractions that contain the lexical operations of real arithmetic. Bhascara, 1150 CE, used only one of Brahmegupta's fractional arithmetics, but we generously credit Bhascara with making this arithmetic consistent. This manoeuvrer makes both of Brahmegupta's total fractional arithmetics consistent. However, Bhascara also introduced an inchoate arithmetic of infinitesimal tuples, which reintroduced Brahmegupta's inconsistency. We suggest that deterministic arithmetics are useful, despite any logical shortcomings. We illustrate this with the IEEE 754 Standard for Floating-Point Arithmetic, which is useful because it succeeds in its aim of specifying a deterministic arithmetic, despite the fact that this arithmetic is non-logical. We conclude that Brahmegupta and Bhascara produced a more logically sound total arithmetic than the IEEE standards committee. We recommend that computer arithmetic is founded on transreal arithmetic, which surpasses the Sanscrit arithmetics in its application to calculus and mathematical physics.

1 Introduction

The invention of the number zero is said to be one of the greatest achievements of the human mind. Today we know whose mind performed this feat. In 628 CE an Indian astronomer, by the name of Brahmegupta, published a treatise on mathematics and astrophysics, parts of which are available in an English translation [16] by Henry Thomas Colebrooke, published in 1817 CE.

Colebrooke [15], [17], [19] was born in London in 1765 CE, the third son of Sir George Colebrooke, 2nd Baronet, Chairman of the East India Company, and Mary Gaynor of Antigua. He was educated at home and at fifteen was adept in classics and mathematics.

In 1782 CE, Colebrooke was appointed to a writership in Calcutta, the most junior clerical position in the East India Company's Civil Service. After postings elsewhere in India, he left the East India Company. In 1801 CE, he returned to Calcutta and was appointed as a judge of the new court of appeal, where he completed the translation from Sanscrit of the *Digest of Hindu Laws,* which had been left unfinished by Sir William Jones. In 1805 CE, he published *A Grammar of the Sanscrit Language* and Lord Wellesley appointed him honorary professor of Hindu Law and Sanscrit at Fort William in Calcutta. In 1807 CE, he was elected President of the *Asiatick Society*, also in Calcutta.

Following the death of his wife, Colebrooke returned to England in 1814 CE, where he played an active role in promoting knowledge. He was co-founder of the *Royal Astronomical Society*. He was present at the inaugural dinner and meeting on January 12, 1820 CE. More significantly for the *Royal Asiatic Society*, he instigated its foundation, with the initial planning meetings taking place at his home in 1823 CE. He presided as Director of the Society until his death in 1837 CE. He was also a Fellow of the *Royal Society* (FRS), a Fellow of the *Royal Society of Edinburgh* (FRSE), and a Fellow of the *Linnean Society of London* (FLS). His collection of Indian plants is maintained in the *Royal Botanic Gardens* at Kew, London.

Today, Colebrooke is recognised as having played a pivotal role in bringing the early advancement of Sanscrit mathematics to Western attention, thereby helping to counterbalance an overly Hellenistic view of the development of mathematics.

Brahmegupta published two treatises in 628 CE. The Ganita assumes familiarity with a decimal place-value arithmetic that existed since ancient times. This used nine numerals corresponding to our 1, 2, 3, 4, 5, 6, 7, 8, 9. Sanscrit mathematicians set out calculations in a tableau, organised in rows and columns, much as we do arithmetic on paper or in a spreadsheet. The absence of a digit was denoted by a blank cell in the tableau. The Sanscrit for "blank" is "sunya." In time, sunya came to mean "numeral zero." But leaving a blank was prone to error, so Sanscrit mathematicians developed the habit of writing a dot (.) or a small circle ($\circ$) to denote a sunya. But a dot was already written above a numeral or variable to denote a negative quantity, so writing a dot in a cell above a numeral or variable was itself ambiguous. In time the circle, $\circ$, became the preferred notation, which is the origin of our numeral zero, 0.

The Sanscrit decimal place-value arithmetic was passed down to us via translation through Arabic, Latin, Italian, and, eventually, English. At each stage, the translators retained the ordering of digits so that in each time period and language, a one-to-one substitution of numerals describes the same number in a decimal expansion and the same bijective substitution of numerals lays out a tableau calculation, such as addition, subtraction, multiplication, and division, in a form that can be read and understood today.

However, the Latin translators described the numerals zero to nine as "Arabic numbers," so that we Europeans lost the connection to Sanscrit mathematics or "Hindu numbers."

The ancient Sanscrit mathematicians had fractions of positive decimal integral numbers. For example, the mixed fraction $i + \frac{n}{d}$, with positive components i, n, and d, was written in a tableau, without writing the plus sign, $+$, or the horizontal bar, -, separating the numerator and denominator of a fraction. Whence, the Sanscrit tableau

$$\begin{array}{cc} i & n \\ & d \end{array}$$

prefigures our mixed fraction $i\frac{n}{d}$. We would like to know if there is a historical chain of evidence that settles the origin of our notation for mixed fractions?

The ancient Sanscrit mathematicians also had some computable irrational numbers, such as square roots, cube roots, and combinations of these.

The ancient Sanscrit mathematicians conceived of negative quantities as being the contrary of positive quantities. They deduced some of the algebraic structure of the addition, subtraction, multiplication, and division of positive and negative quantities by considering, for example, the distance travelled between cities. Assuming that the absolute distance between cities is fixed, but taking, say, the distance to a city as positive and the distance from a city as its contrary, here negative, they deduced some of the algebraic structure of signed numbers by a hypothetical process of armchair reasoning.

In his second treatise, the Cuttaca, Brahmegupta describes and extends an algorithm, known as the Pulverizer, that was invented a century before his time. The Pulverizer was used to calculate orbits of planets and the apparent motion of stars. This algorithm could take some combinations of negative, zero, and positive numbers in its input, execution, and output. Thus, armchair reasoning about signed numbers was tested empirically in astronomy. This testing was carried out for a century before Brahmegupta's time and for many centuries after.

Brahmegupta was an astronomer and astrophysicist, so he was well placed to understand and test earlier arithmetics. In the Cuttaca, he defined the number zero in two ways. Firstly, he introduced the trichotomy of negative, zero, and positive; but he did not relate this to ordering. Brahmegupta did not say that a number can be less than zero. Secondly, Brahmegupta defined zero by giving its algebraic properties in an arithmetic of the addition, subtraction, multiplication, and division of fractions. But his specification of addition has multiple ambiguities, which we now list.

Firstly, Brahmegupta describes addition with different denominators, s_1 and s_2:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{(r_1 \times s_2) + (r_2 \times s_1)}{s_1 \times s_2}$$

Secondly, Brahmegupta describes addition with a common denominators, s:

$$\frac{r_1}{s} + \frac{r_2}{s} = \frac{r_1 + r_2}{s}$$

Thirdly, Brahmegupta seems to say that the above addition with different denominators shall always be used, even where the denominators are common. We call this algorithm *Addition A*.

Fourthly, Brahmegupta seems to say that the above addition with common denominators shall be used when the denominators are, in fact, common, otherwise the above addition with different denominators shall be used when the denominators are, in fact, different. We call this algorithm *Addition B*.

This four-way ambiguity has no effect on the outcome of the addition of finite fractions, but it generally produces different answers when two fractions with a a common denominator of zero are added. Both Brahmegupta and Bhascara discuss such non-finite sums so the four-way ambiguity is critical.

We would like to know whether Sanscrit scholars can dissolve this ambiguity or else confirm that the ambiguity is present in the Sanscrit text?

Thus, Brahmegupta specified two fractional arithmetics: one with the Addition A algorithm and the other with the Addition B algorithm. Both of Brahmegupta's arithmetics are lexically identical to the corresponding operations of rational and hence of real arithmetic, but as they operate on fractions with a zero denominator, both arithmetics exceed real arithmetic. Unfortunately, Brahmegupta made both of his arithmetics inconsistent by defining $\frac{0}{0} = 0$. Nonetheless, when we know which one of Addition A or Addition B is in use, and what the denominator, s, of zero is, in $0 = \frac{0}{s}$, Brahmegupta's arithmetics are uniquely determined for every combination of arguments. We say that a "paraconsistent arithmetic" is uniquely determined, despite contradicting its axioms, whence, under the above conditions, both of Brahmegupta's arithmetics are paraconsistent.

Bhascara was also an astronomer. He published two treatises in 1150 CE. These reflect the structure of Brahmegupta's treatises. Bhascara's Lilavati described decimal arithmetic and his Vija-ganita rehearsed most of Brahmegupta's arithmetic of signed fractions. Bhascara also described and extended the Pulverizer.

Bhascara did not accept Brahmegupta's equality $\frac{0}{0} = 0$, but was silent on the properties of the fraction $\frac{0}{0}$. If we take $\frac{0}{0}$ as any one of positive, negative, or zero, then Bhascara's arithmetic repeats the inconsistencies of Brahmegupta's arithmetic, but these inconsistencies dissolve if we take $\frac{0}{0}$ as an unsigned tetrachotomous number [4]. This choice brings Brahmegupta's arithmetic closer to both real and transreal arithmetic by removing all contradictions.

Bhascara also introduced an inchoate arithmetic of infinitesimal tuples $\langle 0, h \rangle$ for all coefficients h. This prefigured some of the properties of the differential calculus but reintroduced the inconsistency of $\frac{0}{0}$.

In Brahmegupta's and Bhascara's writings, and the works of earlier Sanscrit mathematicians, the properties of associativity, commutativity, distributivity, and the ordering of the positive numbers 1, 2, ... were used implicitly, but they were not acknowledged explicitly. However, later Sanscrit mathematicians did make some of these properties explicit in commentaries on Brahmegupta's and Bhascara's treatises.

The Sanscrit mathematicians made a pivotal contribution to science and technology. Our real and transreal arithmetics, and the systems based on them, are built on the foundation of Sanscrit arithmetic.

We now analyse the totality and paraconsistency of Brahmegupta's and Bhascara's arithmetics of addition, subtraction, multiplication, and division.

1.1 Historical Texts and Authors

Colebrooke [16] translates four main texts, two on arithmetic and two on algebra:

1. Brahmegupta's 628 CE text, Arithmetic (Ganita), begins on Colebrooke's page 277.
2. Brahmegupta's 628 CE text, Algebra (Cuttaca), begins on Colebrooke's page 325.
3. Bhascara's 1150 CE text, Arithmetic (Lilavati), begins on Colebrooke's page 1.
4. Bhascara's 1150 CE text, Algebra (Vija-ganita), begins on Colebrooke's page 129.

These are historically important texts that have been commented on for many centuries up to the present day. Colebrooke draws on many commentaries to validate the four texts and to put them into both their mathematical and historical contexts. The last commentary Colebrooke cites is the Vasana-vartica of 1621 CE.

The authors of the texts and commentaries we use are listed in chronological order, though Crishna Bhatta might be listed too early:

1. Brahmegupta, son of Jishnu, known as Brahmegupta, 628 CE.
2. Chaturveda Prithudaca Swami, known as Chaturveda, after 628 CE and now known to be after about 830 CE, the date of his birth, and before about 890 CE, the date of his death.
3. Bhascara Acharya, known as Bhascara, 1150 CE.

4. Crishna Bhatta, after 1150 CE and before Crishna, below.
5. Scanda Sen Acharya, after 1150 CE and before 1420 CE.
6. Suryadasa, 1541 CE.
7. Ganesa, 1545 CE.
8. Crishna, son of Ballala and pupil of Vishnu, the disciple of Ganesa's nephew Nrisinha, known as Crishna, before 1602 CE.

1.2 Pragmatic Translation

Colebrooke's book [16] was published in English in 1817 CE. The English language has changed since then, but the modern reader should have little difficulty with the generality of the text. There are, however, a few words that require explanation, given in this section, and a few phrases that resist easy analysis, discussed below in the review.

We work from a Google Books copy of the text [16]. This presents a facsimile of the printed text, which we read, together with facilities to search and copy text. The copied text may then be pasted into our text. The search and copy/paste facilities are enabled by Optical Character Recognition (OCR) that was done when the Google Book was compiled. This machine reading of the English text is usually accurate, but the transcription of Sanscrit diacriticals is unreliable. We use the machine-read text but strip it of diacriticals. We hope this is not too troublesome for scholars of Sanscrit. Also, in order to make maximum use of machine transcription, we use Colebrooke's spelling of words rather than editing them. For example, we use Colebrooke's "Sanscrit," not the modern spelling "Sanskrit."

Perhaps Google Books could adopt a mechanism to crowd source the correction of OCR texts by their human readers?

Our objective is to provide a pragmatic translation, into modern English, of the parts of Colebrook's 19th Century English text that deal with the four fundamental operations of arithmetic: addition, subtraction, multiplication, and division. Specifically, we target the technical language of mathematics by using modern mathematical vocabulary and modern mathematical symbols. Thus, we aim to convey the mathematical ideas in Colebrooke's text to the modern reader. We are less concerned with the historical understanding of these ideas. Such historical analysis, while interesting, is both too broad ranging and too fine grained to be undertaken within the confines of Colebrooke's text.

In our translation, we employ the Principle of Generosity. Wherever possible, a phrase is given a coherent mathematical translation. For example, the phrase "they must be thrown together" is rendered as "they must be added." This generous translation can then be checked against Colebrooke's footnotes and any examples of calculations given in the text – though there are special difficulties with the examples, due to the vagaries of Sanscrit mensuration.

Sanscrit mathematicians laid out calculations in a tableau, organised by rows and columns. Negative literal numbers and negative variables, denoted by a single Sanscrit letter or, sometimes, by a short abbreviation of a Sanscrit word, were written with a dot above them, to distinguish them from zero and positive values; but there were no symbols for addition, subtraction, multiplication, division, ordering, bracketing, or equality anywhere in Sanscrit writing! Instead, the original author, commentators, and translators all used an accompanying verbal description of the calculation to interpret the tableau.

Some of the words used to describe spatial relationships within a tableau have standard meanings. For example "above" generally means "numerator" and "below" generally means "denominator." The Sanscrit practice of writing fractions as a numerator over a denominator was passed down to us, via Arab scholars, though we now use a horizontal bar to distinguish the numerator from the denominator. The word "beside" generally refers to an infix calculation, so, in an artificial example, "throw together s beside r" means "$r + s$" and, in an actual example, "zero beside h" means the tuple $\langle h, 0\rangle$, although, for reasons which will become apparent later, we prefer to write this tuple as $\langle 0, h\rangle$.

We also employ the Principle of Least Commitment. For example, when we read of Bhascara's "infinite" quantities, we first check, by wider reading of the text and Colebrooke's footnotes, and by examining calculations, that the generality of non-finite numbers is not intended. Next we check that Bhascara does not mean signed infinities, and only then do we ascribe the meaning of unsigned infinities. In other words, we err on the side of giving conservative mathematical interpretations.

A few English words are problematical for the modern reader, not just because English has changed in the intervening two centuries, but because Colebrooke describes mathematical concepts as they were expressed up to one and a half millennia ago in the ancient language of Sanscrit.

The word "homogeneous" has four meanings, the first three of which are related: (1) "expressed as vulgar fractions;" (2) "expressed as vulgar fractions with a common denominator;" (3) "expressed as vulgar fractions after term flattening;" (4) "identically named algebraic variables." Meanings (1) and (2) echo the ambiguity in Brahmegupta's definitions of addition and subtraction. Meaning (1) leads to Addition A, while meaning (2) leads to Addition B. This ambiguity makes Brahmegupta's arithmetic bifurcate into two arithmetics that generally differ in the sum and difference of two fractions, where both fractions have a common denominator of zero.

Six out of eight occurrences of "nothing" mean "not anything." The single occurrence of "nothing" on Colebrooke's page 136 means "zero" and the first occurrence of "nothing" on Colebrooke's page 137 also means "zero." All occurrences of "cipher" and "nought" mean "zero." But there are four kinds of zero. First, the component zero, 0, may occur as a real numerator or denominator in a fraction. Second, zero may occur as the class $\frac{0}{d} = 0$, where d is a non-zero real denominator. Third, zero may occur as the fraction $\frac{0}{0} = 0$. Fourth, zero may occur as an infinitesimal tuple $\langle 0, h\rangle$, where h is an arbitrary coefficient.

We draw a distinction between the words "finite," "infinite" and "non-finite." For both the Sanscrit authors and Colebrooke, a "finite" number has a bounded positive or negative magnitude that is not zero. For them, zero is a "non-finite" number with no magnitude, whereas "infinite" numbers have an unboundedly large magnitude. When discussing our own work, we use "zero" in its modern meaning of a "finite number with no magnitude." When we discuss Sanscrit text, we take care to remind the reader of the Sanscrit notion that zero is "non-finite but not infinite."

The word "integer" means "integral number" or "whole number." It does not imply modern integer arithmetic, which had not been formalised in Colebrooke's time.

The word "involution" means raising to a square or cubic power. Its inverse, "evolution" means extracting a square or cube root. These operations could be compounded to calculate various powers.

The Sanscrit mathematicians referred to a fraction $\frac{r}{s}$ as an individual fraction for some specific choice of r and s, and as a collection of fractions for many different choices of r_i and s_i, without making any notational distinction between these two uses. In modern terms, they conflated sets and elements of sets. Of course, the Sanscrit mathematicians were writing long before the invention of set theory! We draw this distinction by referring to the implied set as a "class" and by referring to the elements of the class as "fractions" or as Bergstra's syntactic "fracterms," introduced in 2015 CE [10], [12], [13]. When we write $f = \frac{n}{0}$ we mean $f_i \in \{\frac{n_i}{0}\}$ for some non-zero real n_i. But $\{\frac{n_i}{0}\}$ is not an equivalence class in any of Brahmegupta's nor Bhascara's arithmetics, because the choice of a particular fracterm, $\frac{n_i}{0}$, generally makes a difference to the sum and difference of two fracterms with a common denominator of zero. Similarly, $0 = \frac{0}{d}$ means $0_i \in \{\frac{0}{d_i}\}$ for some non-zero real d_i. However, $\{\frac{0}{d_i}\}$ is an equivalence class in all of these fractional arithmetics, as we prove later in Theorem 2.1.

Brahmegupta and Bhascara both wrote their treatises in the form of poems organised in verses, each with a fixed number of syllables following a strict metre. This aided the memorisation of the treatise, but incurred several costs. Firstly, numerical quantities are referred to by different names within a verse, to preserve the metre. This explains why Colebrooke uses different English words – "nothing," "nought," "cipher" – to refer to "zero" within a verse. The word "zero" was not in wide circulation until the twentieth century, so Colebrooke did not use it. Secondly, several mathematical ideas can occur in a verse to use up the available syllables, or a single mathematical idea can be split across several verses where it cannot be expressed in the syllables of one verse. Therefore, there is no direct correlation between verses and mathematical ideas. This is a problem because we want to refer to numbered definitions when presenting modern proofs of Sanscrit arithmetic.

Colebrooke generally numbers each verse within a text, but where a single constellation of coherent mathematical ideas encompasses several verses, he numbers his translation with a run $v_m - v_n$, from the first verse in the constellation v_m, to the last verse v_n. We identify mathematical ideas that interest us, within a text, by an index of the form $v.i$, where v is the verse number and i is the i'th interesting idea within the verse. Where a run of verses occurs, we use the first verse, v_m, and allow i to range over all verses up to and including v_n. Thus, we preserve Colebrooke's verse numbering in our indexes. However, Colebrooke translates four texts and begins the verse numbering anew in each text. In order to obtain a unique index, we prefix the above index with the first letter of the translated name of the Sanscrit text in which a verse appears. Thus: the Ganita is indexed by $Gv.i$, the Cuttaca is indexed by $Cv.i$, the Lilavati is indexed by $Lv.i$, and the Vija-ganita is indexed by $Vv.i$. This allows us to present numbered proofs, in the modern style, with references to numbered definitions.

We also use the indexes to identify how Brahmegupta's ideas map onto Bhascara's ideas. We write a tuple, (x, y), to show which of Brahmegupta's indexed ideas, x, are rehearsed in Bhascara's indexed ideas, y. Thus, when Brahmegupta's C35.5 is rehearsed in Bhascara's L44.4 we write "Rehearsals: (C35.5, L44.4)." We discover, by these means, which parts of Brahmegupta's arithmetic of addition, subtraction, multiplication, and division appear in Bhascara's arithmetic, and we discover which parts of Bhascara's arithmetic are not prefigured by Brahmegupta. For example, in Verse L44, Bhascara adds infinitesimals at L44.5 and L44.6. We write these as "Novelties: L44.5, L44.6."

If our translation from 19th Century to 21st Century English reveals anything of interest, we hope Sanscrit scholars will consider retranslating parts of the original Sanscrit texts.

1.3 Analytical Tools

Brahmegupta's 628 CE text has attracted many commentaries in many languages up to the present day. We find it helpful to adopt fixed variable names in the analysis of Sanscrit mathematics and its commentaries. Brahmegupta and Sanscrit mathematicians before him had access to some of the real numbers, and, as we now know, their work applies to all of the real numbers. Hence, we adopt real variables in the knowledge that our comments and proofs encompass all of the real numbers Brahmegupta and subsequent authors had access to.

Definition 1.1. *Mnemonic – let d_i, n_i, p_i, r_i, s_i be arbitrary real numbers such that: d_i and n_i are non-zero; p_i is positive, whence $-p_i$ is negative; finally r_i and s_i have arbitrary trichotomous signs. Fractions are generally written as $\frac{n_i}{d_i}$ and $\frac{r_i}{s_i}$. A general Sanscrit or Hindu number is written as $h_i = \frac{r_i}{s_i}$.*

This mnemonic makes it easy, firstly, to track numerators and denominators as they move through calculations and, secondly, to compare results that were obtained centuries or even a millennium apart.

Our experience of real arithmetic and modern totalisations of it [4], [5], [7], [8], [10], [11], [12], [13], [14], [18], [20], [21], [23], [24], [25], teaches us that there are common lexical operations in all of the descendants of Sanscrit arithmetic. These can be written in different ways, which are modified in the different totalisations, but we still expect that writing down arithmetics in terms of common lexical units will reveal their relatedness and, perhaps, their developmental history.

We tentatively propose a collection of foundational lexical operations, in addition to the above Definition 1.1, but experience might teach that some other arrangement is more suitable.

We start by giving our lexical units a semantic foundation by adopting real arithmetic as the arithmetic of components. Then we give definitions using these components which may be augmented, modified, or otherwise used to construct or analyse arithmetics descended from Sanscrit arithmetic.

Definition 1.2. *Component Numbers – real arithmetic is the arithmetic of component numbers.*

Definition 1.3. *Fracterm Numbers – fracterms* $\frac{r}{s}$ *are two-tuples of a component numerator* r *and a component denominator* s*. Such fracterms are known as fracterm numbers.*

Definition 1.4. *Tetrachotomy – each number falls into exactly one of four classes: first, less than zero; second, equal to zero; third, greater than zero; fourth, not less than zero and not equal to zero and not greater than zero. Numbers in the first three classes are called trichotomous or ordered numbers. Numbers in the fourth class are called unordered numbers. Numbers in any of the four classes are called tetrachotomous numbers.*

Definition 1.5. *Multiplication - fracterms* $\frac{r_i}{s_i}$ *are multiplied by the operation "×" as:*

$$\frac{r_1}{s_1} \times \frac{r_2}{s_2} = \frac{r_1 \times r_2}{s_1 \times s_2}$$

Definition 1.6. *Division of Components – components* r *and* s *are divided by the operation "÷" as:*

$$r \div s = \frac{r}{s}$$

Definition 1.7. *Reciprocal – a fracterm* $\frac{r}{s}$ *is reciprocated by the operation "*$^{-1}$*" as:*

$$\left(\frac{r}{s}\right)^{-1} = \frac{s}{r}$$

Definition 1.8. *Division of Fracterms – fracterms* $\frac{r_i}{s_i}$ *are divided by the operation "÷" as:*

$$\frac{r_1}{s_1} \div \frac{r_2}{s_2} = \frac{r_1}{s_1} \times \left(\frac{r_2}{s_2}\right)^{-1}$$

Definition 1.9. *Cancellation – a fracterm with a common factor,* c*, in the numerator and denominator,* $\frac{r \times c}{s \times c}$*, may have the factor* c *cancelled, to give* $\frac{r}{s}$*, if and only if* $\frac{c}{c} = 1$*. Thus:*

$$\frac{r \times c}{s \times c} = \frac{r}{s} \times \frac{c}{c} = \frac{r}{s} \times 1 = \frac{r}{s}$$

Definition 1.10. *Addition With Any Denominators – fracterms* $\frac{r_i}{s_i}$*, with any denominators* s_i*, are added by the operation "+" as:*

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{(r_1 \times s_2) + (r_2 \times s_1)}{s_1 \times s_2}$$

Definition 1.11. *Addition With a Common Denominator – addition of fracterms* $\frac{r_i}{s}$*, with a common denominator, s, are added by the operation "+" as:*

$$\frac{r_1}{s} + \frac{r_2}{s} = \frac{r_1 + r_2}{s}$$

Definition 1.12. *Subtraction – fracterms* $\frac{r_i}{s_i}$ *are subtracted by the operation "−" as:*

$$\frac{r_1}{s_1} - \frac{r_2}{s_2} = \frac{r_1}{s_1} + \frac{-r_2}{s_2}$$

Let us stress that we expect these foundational definitions to be used in different ways and to be modified in the analysis and construction of different partial and total descendants of Sanscrit arithmetic. Our lexical definitions are tools for analysis and construction, not prescriptions of how any particular arithmetic should be arranged.

1.4 Brahmegupta's and Bhascara's Arithmetics

We say that Brahmegupta's arithmetic is given by: operation G2.2A, G2.3A, G2.2B, G2.3B, G3.2, G4.2; sign convention C31 in Subsection 2.2.2, C32 in Subsection 2.2.3, C34 in Subsection 2.2.4, C35 in Subsection 2.2.5; Observation G2.4; and Theorem 2.1.

We say that Bhascara's arithmetic of infinitesimal tuples is given in Definition L44.8 and L44.9.

We say that Bhascara's identification of unsigned infinities is given in Definition V14.1 and V14.2.

2 Review

Colebrooke's book [16] is a history of mathematics. The preface, comprising 84 pages, discusses Sanscrit mathematics and how it was communicated to Europe via Arabia, attracting many commentaries during centuries of diffusion. Many of the authors and their commentators were astronomers, who included astronomical data in their texts. Colebrooke cites astronomers, contemporary to him, who used these data to compute approximate dates for the texts, taking into account the relative inaccuracy of the early observations. Colebrooke and others cross-checked these astronomical dates with historical records to establish firm dates. Several calenders were in use in India, which Colebrooke converted to the Christian era and which we render as the Common Era, CE. As New Year's Day differs in the Indian and Christian calenders, the actual date, known from historical documents only within an Indian year, falls on one of two possible Common Era years. Colebrooke used the earlier year, so that, for example, Brahmegupta's texts of 628 CE might actually have been published in 629 CE, and similarly for all of Colebrooke's dates.

On page ii of the preface, Colebrooke laments the late arrival of Sanscrit mathematics in Europe.

In the actual advanced condition of the analytic art, it is not hoped, that this version of ancient Sanscrit treatises on Algebra, Arithmetic, and Mensuration, will add to the resources of the art, and throw new light on mathematical science, in any other respect, than as concerns its history. Yet the remark may not seem inapposite, that had an earlier version of these treatises been completed, had they been translated and given to the public, when the notice of mathematicians was first drawn to the attainments of the Hindus in astronomy and in sciences connected with it, some addition would have been then made to the means and resources of Algebra for the general solution of problems by methods which have been reinvented, or have been perfected, in the last age.

The irony of this comment is that Colebrooke did not notice that Brahmegupta's and Bhascara's arithmetics totalise the arithmetic of Colebrooke's era. A similar totalisation was first perfected by Suppes in 1957 [25], some one hundred and forty years after Colebrook.

The main body of Colebrooke's book runs to 378 pages. It presents his translations of four historically significant Sanscrit texts on arithmetic and algebra, written by Brahmegupta and Bhascara. We review the short sections of these texts that deal with the four fundamental operations of arithmetic: addition, subtraction, multiplication, and division. Our objective is to discover to what extent these are total consistent arithmetics.

2.1 Brahmegupta's 628 CE Ganita

We are expert in total systems. We began our research by reading the whole of Colebrooke's book [16], including all of the footnotes, and by introducing ourselves to Sanscrit mathematics and poetry, as well as to the Hindu religion. We read works on the history of mathematics in many parts of the world and made a brief study of the etymology of mathematical terms. We now analyse the text.

The four fundamental operations of arithmetic – addition, subtraction, multiplication, and division – are set out in verses G2 - G4, with footnotes by the commentator Chaturveda Prithudaca Swami, who is now known to have died about 890 CE. The remainder of the verses in the Ganeta are of no interest to us.

2.1.1 Verse G1

The first verse, G1, of Brahmegupta's 628 CE Ganita, which we reproduce without footnotes, sets out the mathematical preparation needed of the reader to understand his text.

We set Colebrooke's text in italics, as we did in the Review, Section 2, to show that it is a direct quotation from his book [16].

G1 *He, who distinctly and severally knows addition and the rest of the twenty logistics, and the eight determinations including measurement by shadow, is a mathematician.*

Our reader will need some knowledge of rational arithmetic and must have read the above sections of our paper.

2.1.2 Verse G2

The second verse, which we reproduce with Chaturveda's comments, describes addition and subtraction.

G2 *Quantities, as well numerators as denominators, being multiplied by the opposite denominator, are reduced to a common denomination. In addition, the numerators are to be united.*[1] *In subtraction, their difference is to be taken.*[2]

The phrase "as well numerators as denominators" is opaque to the modern reader. This phrase uses an old construction, "as well X as Y" which means "both X and Y." Now the phrase "Quantities, both numerators and denominators" is readable, but it is incoherent in the context of the rest of the verse, because "both numerators and denominators" refers to individual real numbers, whereas the verse refers to fractions. That is to say, the "Quantities" are "Fractions." Sanscrit has words for "fractions" so why is the gloss "both numerators and denominators" needed? The verse describes vulgar fractions, but Sanscrit does not have words to distinguish vulgar from mixed fractions, or proper from improper vulgar fractions. The purpose of the gloss is to indicate that vulgar fractions are meant, regardless of whether they are proper or improper, as may be conveyed by our phrase, "Vulgar fractions, composed only of a numerator and a denominator."

The verse begins at G2.1 with a description of how vulgar fractions are reduced to a common denominator. Then G2.2 describes addition and G2.3 describes subtraction. We now translate Colebrooke's verse G2 and insert our indexes in Latin braces as {Gv.i}.

G2 {G2.1} Fracterms $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$, composed only of a numerator, r_i, and a denominator, s_i, being multiplied by the opposite denominator, giving $\frac{r_1 s_2}{r_1}$ and $\frac{r_2 s_1}{s_2}$, are reduced to a common denominator as $\frac{r_1 s_2}{s_1 s_2}$ and $\frac{r_2 s_1}{r_1 s_2}$. {G2.2} In addition, the numerators are to be summed as $\frac{r_1 s_2 + r_2 s_1}{s_1 s_2}$. {G2.3} In subtraction, their difference is to be taken, as $\frac{r_1 s_2 - r_2 s_1}{s_1 s_2}$.

There is a fundamental ambiguity in this verse. Addition A: G2.1 is part of the definition of the addition and subtraction of vulgar fractions, such that multiplication by opposite denominators must always be calculated, even when the denominators are already common. Addition B: G2.1 is a conditional preconditioning step that must be applied when fractions have different denominators, but which must not be applied when fractions already have a common denominator. This ambiguity makes no difference to sums and differences of finite fractions, where we use "finite" in its modern sense, which implies that denominators are non-zero, but it does make a difference to sums and differences of infinities and the fracterm $\frac{0}{0}$ that have a common denominator of zero.

It is unclear which Addition Brahmegupta used, but Brahmegupta's commentators, Chaturveda and Scanda Sen Acharya, along with Bhascara and his commentator, Crishna, used Addition B.

Chaturveda's first footnote reads: [1] *SCANDA-SEN-ACHARYA, who has exhibited addition by a rule for the summation of series of the arithmeticals, has done so to show the figure of sums; and he has separately treated of figurate quantity (cshetra-rasi), to show the area of such figure in an oblong. But, in this work, addition being the subject, sum is taught; and the author will teach its figure by a rule for the summation of series (§19). In this place, however, sum and difference of quantities having like denominators are shown: and that is fit.*

For our purposes, only the last sentence is important: *In this place, however, sum and difference of quantities having like denominators are shown: and that is fit.* That is, Chaturveda and Scanda Sen Acharya use Addition B.

Chaturveda's second footnote on verse G2 confirms that he used Addition B. He writes: [2] *Example of addition: What is the sum of one and a third, one and a half, one and a sixth part, and the integer three, added together?*

Statement: $1\frac{1}{3}\ 1\frac{1}{2}\ 1\frac{1}{6}\ 3$. *Or reduced* $\frac{4}{3}\ \frac{3}{2}\ \frac{7}{6}\ \frac{3}{1}$

The numerator and denominator of the first term being multiplied by the denominator of the second, 2, and those of the second by that of the first, 3, they are reduced to the same denominator ($\frac{8}{6}\ \frac{9}{6}$*; and, uniting the numerators,* $\frac{17}{6}$*). With the third term no such operation can be, since the denominator is the same: union of the numerators is alone to be made;* $\frac{24}{6}$*, which abridged is) 4. So with the fourth term: and the addition being completed, the sum is 7. Subtraction is to be performed in a similar manner; and the converse of the same example may serve.*

G2.4 Observation: No later than about 890 CE, Chaturveda writes the integer 3 as the fraction $\frac{3}{1}$. We assume it was the practice to write an integer, i, as the fraction, $\frac{i}{1}$. This is confirmed in Bhascara, 1150 CE, in Verse L36: *Unity is put denominator of a quantity which has no divisor.*

In Chaturveda's example, we see that mixed fractions are reduced to vulgar fractions, in the *Statement,* before they are added. That is, any vulgar fractions can be added, not just those with a common denominator, but if they do have a common denominator, then Addition B is compulsory. Chaturveda insists that *...no such operation [as reducing to a common denominator] can be [performed], since the denominator is [already] the same ...* Chaturveda says that a similar process is used with subtraction.

The reader may now compare our statement of addition, in Definition G2.2B, below, with Chaturveda's example, which we render as follows.

$$\begin{aligned}
1\frac{1}{3}+1\frac{1}{2}+1\frac{1}{6}+3 &= \frac{4}{3}+\frac{3}{2}+\frac{7}{6}+\frac{3}{1}\\
&= \frac{4\times 2}{6}+\frac{3\times 3}{6}+\frac{7}{6}+\frac{3}{1}\\
&= \frac{8}{6}+\frac{9}{6}+\frac{7}{6}+\frac{3}{1}\\
&= \frac{17}{6}+\frac{7}{6}+\frac{3}{1}\\
&= \frac{24}{6}+\frac{3}{1}\\
&= 4+3\\
&= 7
\end{aligned}$$

Notice that Chaturveda does not add $\frac{17}{6}+\frac{7}{6}$ by computing $\frac{17\times 6}{6\times 6}+\frac{7\times 6}{6\times 6}=\frac{102}{36}+\frac{42}{36}$. Recall that he says, deleting an erroneous right parenthesis, which may be Colebrooke's fault: *. . . no such operation can be, since the denominator is the same: union of the numerators is alone to be made . . .* Again we see that Chaturveda uses Addition B.

We now define addition and subtraction in a form we can use in modern Proofs. G2.2A and G2.3A form the entirety of addition and subtraction according to Addition A. In Addition B, addition and subtraction of fracterms with a common denominator are given by G2.2B and G2.3B, otherwise, when the denominators are different, G2.2A and G2.3A apply.

G2.2A Definition: in Addition A, the addition, +, of fracterms $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ is given by:

$$\frac{r_1}{s_1}+\frac{r_2}{s_2}=\frac{r_1s_2+r_2s_1}{s_1s_2}$$

This addition is also used, in Addition B, when the denominators are different.

Notice that Brahmegupta's G2.2A is implemented by our lexical operation in Definition 1.10. Therefore, G2.2A is lexically consistent with rational and real arithmetic.

G2.3A Definition: in Addition A, the subtraction, −, of fracterms $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ is given by:

$$\frac{r_1}{s_1}-\frac{r_2}{s_2}=\frac{r_1s_2-r_2s_1}{s_1s_2}$$

This subtraction is also used, in Addition B, when the denominators are different.

Notice that Brahmegupta's G2.3A is implemented by our lexical operations in Definition 1.12, 1.10. Therefore, G2.3A is lexically consistent with rational and real arithmetic.

G2.2B Definition: in Addition B only, the addition, $+$, of fracterms $\frac{r_1}{s}$ and $\frac{r_2}{s}$, with a common denominator, s, is given by:

$$\frac{r_1}{s} + \frac{r_2}{s} = \frac{r_1 + r_2}{s}$$

Otherwise, when the denominators are different, the addition is performed by G2.2A.

Notice that Brahmegupta's G2.2B is implemented by our lexical operation in Definition 1.11. Therefore, G2.2B is lexically consistent with rational and real arithmetic.

G2.3B Definition: in Addition B only, the subtraction, $-$, of fracterms $\frac{r_1}{s}$ and $\frac{r_2}{s}$, with a common denominator, s, is given by:

$$\frac{r_1}{s} - \frac{r_2}{s} = \frac{r_1 - r_2}{s}$$

Otherwise, when the denominators are different, the subtraction is performed by G2.2A.

Notice that Brahmegupta's G2.3B is implemented by our lexical operations in Definition 1.12, 1.11. Therefore, G2.3B is lexically consistent with rational and real arithmetic.

These definitions are lexically total, because we can always write down a sum or difference of terms but, since Brahmegupta does not yet have the additive inverse, zero, he cannot evaluate $r - r$, nor can he calculate the interaction of positive and negative terms, whence the definitions are semantically partial and have only partial associativity, partial commutativity, and partial distributivity.

At this stage, Brahmegupta does not yet have sufficient algebraic structure to convert a subtraction into an addition with the negated subtrahend. This is provided by Brahmegupta's C31.4 in Subsection 2.2.2 and Bhascara's Verse V3.

2.1.3 Verse G3

The third verse, which we reproduce with one of Chaturveda's comments, describes multiplication.

G3 *Integers are multiplied by the denominators and have the numerators added. The product of the numerators, divided by the product of the denominators, is multiplication*[3] *of two or of many terms.*[4]

Chaturveda's third footnote is of no interest to us, but his fourth footnote explains the first sentence, *Integers are multiplied by the denominators and have the numerators added.*

Chaturveda's fourth footnote on verse G3 reads: [4] *Example: Say quickly what is the area of an oblong, in which the side is ten and a half, and the upright seventy sixths.*

Statement: $10\frac{1}{2}$ $11\frac{4}{6}$*. Multiplying the integers by the denominators, adding the numerators, and abridging, the two quantities become* $\frac{21}{2}$ *and* $\frac{35}{3}$*. From the product of the numerators 735, divided by the product of* *the denominators 6, the quotient obtained is* $122\frac{1}{2}$. *It is the area of the oblong.*

Others here exhibit an example of the rule of three terms, making unity stand for the argument or first term. For instance, if one pala of pepper be bought for six and a half panas, what is the price of twenty-six palas? Answer : 169 *panas.]*

In this comment, we see that the mixed fraction, $10\frac{1}{2}$, is reduced to the vulgar fraction, $\frac{21}{2}$, by multiplying the integer, 10, by the denominator, 2, to give $10 \times 2 = 20$, and then by adding the numerator, 1, to give $20 + 1 = 21$, as the numerator of the vulgar fraction $\frac{21}{2}$. Chaturveda then reduces the mixed fraction $11\frac{4}{5}$ to the vulgar fraction $\frac{35}{3}$, after it is *abridged* by cancelling the highest common factor. Notice that $\frac{21}{2}$ and $\frac{35}{3}$ are both vulgar fractions, but they do not have a common denominator.

In summary, verse G3 begins at G3.1 with a description of the precondition that quantities are vulgar fractions. Then G3.2 describes multiplication. We now translate Colebrooke's verse G3 and insert our indexes.

G3 {G3.1} Integers, i, in mixed fractions, $i\frac{n}{d}$, are multiplied by the denominators, d, as id, and have the numerators added, as $id + n$, to convert mixed fractions, $i\frac{n}{d}$, to fracterms $\frac{id+n}{d} = \frac{r_i}{s_i}$. {G3.2} Then the product of the numerators, $r_1 \times r_2$, divided by the product of the denominators, $s_1 \times s_2$, is multiplication of two $\frac{s_1 \times s_2}{r_1 \times r_2}$ or of many $\frac{s_1 \times s_2 \times \ldots s_i}{r_1 \times r_2 \times \ldots r_i}$ fracterms.

We now define multiplication in a form we can use in modern proofs.

G3.2 Definition: the multiplication, $\times$, of fracterms $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ is given by:

$$\frac{r_1}{s_1} \times \frac{r_2}{s_2} = \frac{r_1 \times r_2}{s_1 \times s_2}$$

Notice that Brahmegupta's G3.2 is implemented by our lexical operation in Definition 1.5. Therefore, G3.2 is lexically consistent with rational and real arithmetic.

At this stage, G3.2 is semantically partial because Brahmegupta does not yet have sufficient algebraic structure to describe the interaction of positive and negative multiplicands and zero.

2.1.4 Verse G4

The fourth verse, which we reproduce with Chaturveda's comments and Colebrooke's addendum, describes division.

G4 *Both terms being rendered homogeneous,*[5] *the denominator and numerator of the divisor are transposed: and then the denominator of the dividend is multiplied by the [new] denominator; and its numerator, by the [new] numerator. Thus division*[1] *[is performed.]*

Chaturveda's fifth footnote on verse G4 reads: [5] *The method of rendering homogeneous has been delivered in the foregoing rule (§3) "Integers are multiplied by the denominators," &c.*

In this comment "(§3)" refers to the verse we call "G3" and "&c" means "etcetera." Here, Chaturveda refers to the reduction of a mixed fraction to a vulgar fraction, as we discussed above, in Verses G2 and G3. Division is then performed on any vulgar fractions.

Colebrooke's addendum to this comment reads: *It is reduction to the form of an improper fraction.* This is correct, in so far as it agrees with our Principle of Least Commitment, but it misses the mathematical point: Brahmegupta's verbal description of addition, subtraction, and multiplication is consistent with rational arithmetic, when the quantities involved are vulgar fractions, regardless of whether they are proper or improper. Employing the Principle of Generosity, we read Brahmegupta as introducing a canonical form – of vulgar fractions – and then giving a lexical account of the operations of rational arithmetic. To coin an English phrase, Colebrooke cannot see the wood for the trees!

We now translate Colebrooke's Verse G4 and insert our indexes.

G4 {G4.1} Both terms being rendered homogeneous, as $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$, {G4.2} the denominator and numerator of the divisor are transposed, giving $\frac{s_2}{r_2}$, and then the denominator of the dividend, s_1, is multiplied by the new denominator, r_2; and its numerator, r_1, by the new numerator, s_2. Thus, division is performed as $\frac{r_1 \times s_2}{s_1 \times r_2}$.

Chaturveda's first footnote on Verse G4 says: [1] *Example: In a rectangle, the area of which is given, a hundred and twenty-two and a half; and the side, ten and a half; tell the upright.*

Statement: $122\frac{1}{2}$ $10\frac{1}{2}$. *Reduced to homogeneous form* $\frac{245}{2}$ $\frac{21}{2}$

Here, the side is divisor. Its denominator and numerator are transposed $\frac{2}{21}$. *The numerator of the dividend, multiplied by this numerator, becomes 490; and the denominator of the dividend, taken into the denominator, makes 42. The one, divided by the other, gives the quotient* $11\frac{1}{2}$. *It is the upright.*

Some in this place also introduce an example of the rule of three terms. Thus "A king gave to ten principal priests a hundred thousand pieces of money, together with a third of one piece. What was the wealth that accrued to one?"

In this comment, the mixed fractions $122\frac{1}{2}$ and $10\frac{1}{2}$ are reduced to improper vulgar fractions, respectively, $\frac{245}{2}$ and $\frac{21}{2}$. This is the same preconditioning as in Verses G2 and G3. But here, too, the calculation that follows holds for all vulgar fractions, regardless of whether they are proper or improper.

Let us check the calculation of the upright using definition G4.2 as follows.

$$122\frac{1}{2} \div 10\frac{1}{2} = \frac{245}{2} \div \frac{21}{2} = \frac{245}{2} \times \frac{2}{21} = \frac{490}{42} = 11 + \frac{28}{42} = 11\frac{2}{3}$$

Thus, we find a mistake. Colebrooke says that Chaturveda says the upright is $11\frac{1}{2}$, but we calculate the upright as $11\frac{2}{3}$, which we verify by decimal calculation and cross-checking in decimal: $10.5 \times 11.5 = 120.75 \neq 122.5$. The example is wrong, but who is at fault – Colebrooke or Chaturveda?

We notice that the area of $122\frac{1}{2}$ and the side of $10\frac{1}{2}$ also occur in Chaturveda's fourth footnote on Verse G3, presented above. In that place

Chaturveda correctly gives the upright as $11\frac{4}{6}$ which is equal to our calculated $11\frac{2}{3}$. We judge it more likely that Colebrooke is in error.

Whatever the case may be, we employ the Principle of Generosity and say that while division has been correctly described, Colebrooke or Chaturveda made an arithmetical slip.

We now define division in a form we can use in modern proofs.

G4.2 Definition: the division, $\div$, of fracterms $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ is given by:

$$\frac{r_1}{s_1} \div \frac{r_2}{s_2} = \frac{r_1 \times s_2}{s_1 \times r_2}$$

Notice that Brahmegupta's G4.2 is implemented by our lexical operations in Definition 1.7, 1.8. Therefore, G4.2 is lexically consistent with rational and real arithmetic.

At this stage, G4.2 is semantically partial for the same reasons as multiplication.

2.1.5 Summary of the Ganita

In the Ganita, Brahmegupta assumes prior knowledge of the addition, subtraction, multiplication, and division of positive decimal integral numbers. Division may be exact or with remainder. Brahmegupta, his predecessors and successors, up to the seventeenth century, had an implicit understanding of associativity, commutativity, distributivity, equality, and the ordering of positive integral numbers. Brahmegupta's successors gave an explicit account of some of these properties.

Our main finding is that in 628 CE, Brahmegupta gave verbal descriptions of the operations of addition, subtraction, multiplication, and division of fractions, that are lexically identical to the corresponding operations of rational, and hence real arithmetic. An arithmetical slip not withstanding, these lexical operations are confirmed by numerical examples given by Chaturveda no later than about 890 CE.

2.2 Brahmegupta's 628 CE Cuttaca

Brahmegupta's Cuttaca begins with his refinement of the Pulverizer, in Verses C1 - C30. He then introduces the algebraic properties of signed numbers and the number zero in Verses C31 - C36. It is these algebraic properties that establish zero as a number, independently of the notation used to represent zero and independently of zero's use as an empty place holder in the decimal position-value system that had a long development in India, including in Brahmegupta's time. The remainder of the 103 verses of the Cuttaca are of no interest to us.

2.2.1 The Pulverizer

The Pulverizer is described in Verses C1 - C30 of Colebrooke's book [16]. The Pulverizer solves certain Diophantine equations by using repeated division with remainder. In Brahmegupta's refinement, the number zero, not just a placeholder for the absence of a counting number, and both positive and negative integral numbers, can occur in the input, execution, and output of the algorithm, but only in limited combinations.

The algorithm imposes some algebraic structure on zero and negative integers. For example, in the working of Verse C7 we learn that where c_i are counting numbers, accessing positive integers, the algorithm imposes: $c \times 0 = 0$; $0 - c = -c$, where $-c$ is the opposite of c, accessing a negative integer; finally, a positive counting number, c_1, accessing a positive integer, divided by a negative counting number, $-c_2$, accessing a negative integer, has a remainder which is a negative counting number, $-c_3$, accessing a negative integer. These properties are subsumed in Brahmegupta's explicit introduction of the sign conventions in Verses C31 - C36, below.

We would like to know all of the algebraic properties that were implemented in the Pulverizer before Brahmegupta's time, so that we could properly assess his contribution.

The Pulverizer is executed in a tableau, where a remainder, n, is written above a divisor, d. This vertical arrangement of numbers is lexically indistinguishable from the fraction $\frac{n}{d}$. Colebrooke uses this fractional form in his translation, but stops short of writing correct but strange looking equations such as $3 \div 2 = \frac{1}{2}$.

2.2.2 Verse C31

The thirty-first verse has no comments. It describes the sign convention of addition.

C31 *Rule for addition of affirmative and negative quantities and cipher: §19. The sum of two affirmative quantities is affirmative; of two negative is negative; of an affirmative and a negative is their difference; or, if they be equal, nought. The sum of cipher and negative is negative; of affirmative and nought is positive; of two ciphers is cipher.*

Before Brahmegupta, Sanscrit mathematicians could calculate with the counting numbers in a way that produces the same result as the corresponding calculation with the positive integers. We make the generous assumption that, in Verses C31 - C36, Brahmegupta is describing how signs are affixed to calculated magnitudes. This interpretation is shared by Crishna in his comment on multiplication in Footnote 1 on Verse V7: *The sign only of the product is taught. All the operations upon the numbers are the same which were shown in simple arithmetic (Lila. § 14-16).*

C31 {C31.1} Rule for addition of positive (+ve) and negative (-ve) magnitudes and zero. {C31.2} The sum of two positive magnitudes is positive: (+ve) + (+ve) = (+ve). {C31.3} The sum of two negative magnitudes is negative: (-ve) + (-ve) = (-ve). {C31.4} The sum, $p + n$, of a positive number, $p = |p|$, and a negative number, $n = -|n|$, is their difference: $p - |n|$. {C31.5} The sum of a positive and a negative number, with equal magnitudes, n, is zero: $n + (-n) = 0$. {C31.6} The sum of zero and a negative magnitude is negative: 0 + (-ve) = (-ve). {C31.7} The sum of a positive magnitude and zero is positive: (+ve) + 0 = (+ve). {C31.8} The sum of two zeros is zero: $0 + 0 = 0$.

Notice that Verse C31 assumes commutativity. This is also the modern practice – except when discussing non-commutative algebras!

Multiplicative commutativity is confirmed in Colebrooke's translation of Bhascara, 1150 CE, Verse V72, second footnote, by CRISHNA BHATTA, of uncertain date: *...there is no difference whether quantities be multiplicator or multiplicand to each other.*

Thereafter, additive commutativity of the addition of integers and rational fractions is deducible from Verse V7, second footnote, by Colebrooke, *Multiplication, as explained by the commentators,* is a sort of addition resting on repetition of the multiplicand as many times as is the number of the multiplicator.* Colebrooke names the commentators: * SURYADASA, 1541 CE; GANESA, 1545 CE; CRISHNA-BHATTA of uncertain date.

C31.1 is a statement that sets out the scope of addition. This scope is the trichotomy of numbers in the mutually exclusive classes of negative, zero, and positive. However, as we shall see, trichotomy is not enforced on all of Brahmegupta's numbers, nor is trichotomy rewritten in terms of ordering. Brahmegupta does not say numbers may be less than zero, equal to zero, or greater than zero. Compare this with our definition of tetrachotomy in Definition 1.4.

C31.4 uses difference, that is it uses subtraction, which is defined in Verses C32 and C33. Thus, C31.4 is a forward reference to those verses. Therefore, we must read all three verses, C31 - C33, before we can give an adequate account of addition and subtraction. When this is done, C31.4 justifies rewriting subtraction as addition with the negated subtrahend: $r - s = r + (-s)$. This rewrite is confirmed by Bhascara, 1150 CE, in Verse V3, below.

C31.5 defines the additive inverse of non-zero numbers: $n - n = 0$. This is extended to the additive inverse of zero at C32.5 next.

2.2.3 Verse C32 - C33

The thirty-second and thirty-third verses have no comments. They describe the sign convention of subtraction.

C32 - C33 *Rule for subtraction : §20-21. The less is to be taken from the greater, positive from positive; negative from negative. When the greater, however, is subtracted from the less, the difference is reversed. Negative, taken from cipher, becomes positive; and affirmative, becomes negative. Negative, less cipher, is negative; positive, is positive; cipher, nought. When affirmative is to be subtracted from negative, and negative from affirmative, they must be thrown together.*

We translate these verses as follows. Recall that our indexes run from the earliest verse, here C32.

C32 - C33 {C32.1} Rule for subtraction. In the examples, v, w, x, y, z are positive real numbers and z is greater than y. {C32.2} If the lesser magnitude number is taken from the greater magnitude number then the result has the same sign when positive is taken from positive, $x = z - y$, or negative is taken from negative, $-x = -z - (-y)$. {C32.3} However, if the greater magnitude number is taken from the lesser magnitude number then the result has the opposite sign when positive is taken from positive, $-x = y - z$, or negative is taken from negative, $x = -y - (-z)$. {C32.4} Negative taken from zero becomes positive, $0 - (-x) = x$, and positive taken from zero becomes negative, $0 - x = -x$. {C32.5} Zero taken from zero is zero, $0 - 0 = 0$. {C32.6} When positive is subtracted from negative, and negative from positive, their magnitudes are added and the result has the sign of the left-hand argument, respectively, as $(-v) - w = -(v + w)$ and $v - (-w) = v + w$.

The forward reference from C31.4 is now resolved, giving us an account of Brahmegupta's addition and subtraction. These operations are lexically total, but are semantically partial, because they cannot immediately evaluate the non-trichotomous classes introduced in Verses C35 - C36.

Our translation of C32.6 is more generous than usual, but something of this sort is needed to make mathematical sense of the original.

2.2.4 Verse C34

The thirty-fourth verse has no comments. It describes the sign convention of multiplication.

C34 *Rule for multiplication: § 22. The product of a negative quantity and an affirmative is negative; of two negative, is positive; of two affirmative, is affirmative. The product of cipher and negative, or of cipher and affirmative, is nought; of two ciphers, is cipher.*

C34 {C34.1} Rule for multiplication. {C34.2} The product of a negative number and a positive number is negative. {C34.3} The product of two negative numbers is positive and the product of two positive numbers is positive. {C34.4} The product of zero and a negative number, and of zero and a positive number, is zero. {C34.5} The product of two zeros is zero.

Multiplication is lexically total, but is semantically partial, because it cannot evaluate the signs of the non-trichotomous classes introduced in Verses C35 - C36, discussed next.

We find it slightly odd that there are no comments on Verses C31 - C34. Perhaps the arithmetic was so well known that no comments were deemed necessary? This might be settled by wider historical research.

2.2.5 Verse C35 - C36

The thirty-fifth and thirty-sixth verses have comments by Colebrooke. These verses describe the sign convention of division.

C35 - C36 *Rule for division: § 23-24. Positive, divided by positive, or negative by negative, is affirmative. Cipher, divided by cipher, is nought. Positive, divided by negative, is negative. Negative, divided by affirmative, is negative. Positive, or negative, divided by cipher, is* *a fraction with that for denominator:*[1] *or cipher divided by negative or affirmative.*[2]

Colebrooke gives a continuation of Verse C36, but the continuation does not add anything of interest to us.

The last sentence of Verses C35 - C36 is somewhat opaque to the modern reader: *Positive, or negative, divided by cipher, is a fraction with that for denominator:*[1] *or cipher divided by negative or affirmative.*[2]

Firstly, what does *that* mean? Colebrooke provides a footnote: [1] *Tachchheda, having that for denominator: having, in this instance, cipher for denominator, to a finite quantity for numerator. See Vij.-gan. § 16.* But notice that Colebrooke does not regard zero as being finite, whereas the modern mathematician does take zero as finite. To avoid confusion, we avoid using the word *finite* in our translation, thereby keeping closer to Brahemgupta's original than Colebrooke's comment. So we translate *Positive, or negative, divided by cipher, is a fraction with that for denominator* as "A positive or negative number, $\pm p$, divided by zero, is a fraction of that number, $\pm p$, with zero for denominator: $\frac{\pm p}{0}$."

Colebrooke says, *See Vij.-gan. § 16.*, which is a reference to the verse we call V16. Verse V16 confirms that there are fractions that have a zero denominator, *In this quantity consisting of that which has cipher for its divisor, ...*, but the rest of the verse is opaque. We now suppose we have the correct understanding of *that.*

But we have a harder problem. What does *or cipher divided by negative or affirmative.*[2] mean? Colebrooke provides a footnote: [2] *Is in like manner expressed by a fraction having a finite denominator to a cipher for numerator.* But, again, we avoid Colebrooke's word *finite.* We read this as "or cipher divided by negative or affirmative is in like manner expressed by a fraction having a positive or negative denominator to a cipher for numerator."

Drawing all of this together, we translate Colebrooke's Verse C35 - C36 as follows.

C35 - C36 {C35.1} Rule for division. {C35.2} Positive divided by positive or negative divided by negative is positive. {C35.3} Zero divided by zero is zero. {C35.4} Positive divided by negative is negative. Negative divided by positive is negative. {C35.5} A positive or negative number, $\pm p \neq 0$, divided by zero, is the fraction $\frac{\pm p}{0}$ {C35.6} or zero divided by $\pm p$ is the fraction $\frac{0}{\pm p}$.

Note that C35.3 says $\frac{0}{0} = 0$, C35.5 says that the class $\frac{n}{0}$ exists for all non-zero real n, and C35.6 says that the class $\frac{0}{d}$ exists for all non-zero real d.

We already have enough of Brahmegupta's arithmetic to present a theorem. We state the theorem in terms of an arbitrary non-zero real numerator, n, and an arbitrary non-zero real denominator, d, which includes all of the numerators and denominators accessible to Brahmegupta and subsequent authors.

Our theorem relies on Observation G2.4, which was prefigured by Chaturveda no later than 890 CE, and was stated in its general form by Bhascara in 1150 CE. It is not clear whether Brahmegupta was aware of the equality $r = \frac{r}{1}$?

Theorem 2.1. $0 = \frac{0}{d}$ *for all non-zero real* d.

Proof. Let n and d be arbitrary non-zero real numbers. Then:

$$
\begin{aligned}
0 &= \frac{0}{1} \text{ by Observation G2.4} \\
&= \frac{0}{1} \times \frac{n}{d} \text{ by C34.4 in Subsection 2.2.4} \\
&= \frac{0 \times n}{1 \times d} \text{ by Definition G3.2} \\
&= \frac{0}{d} \text{ by C34.2, C34.3, C34.4 in Subsection 2.2.4}
\end{aligned}
$$

□

Thus, the class $\frac{0}{d} = 0$, for all non-zero real d, is trichotomous; but the class $\frac{n}{0}$, for all non-zero real n, remains problematical at this stage, because its magnitudes and signs are not yet clear. Bhascara considered these problems in 1150 CE, as we see below in Definition V14.1 and V14.2. At this stage, Brahmegupta's division is lexically total. When the magnitudes and signs of $\frac{n}{0}$ are resolved, both multiplication and division become semantically total, though $\frac{0}{0} = 0$ remains inconsistent with Brahmegupta's sign conventions, most of which Bhascara accepted.

2.2.6 Summary of the Cuttaca

The algorithm, known as the Pulverizer, solves certain Diophantine equations, using repeated division with remainder. It was known, in some form, around a century before Brahmegupta. The algorithm can take some limited combinations of positive, negative, and zero integral arguments. It operates on all of these internally, in limited ways, before outputting trichotomously signed results. Brahmegupta both extended the Pulverizer and adopted its internal handling of negative, zero, and positive numbers as properties of his arithmetics.

Brahmegupta stated the trichotomy of negative, zero, and positive numbers, but he did not enforce trichotomy, nor did he relate it to ordering. In particular, Brahmegupta did not say that negative numbers are less than zero.

Brahmegupta asserted the existence of fractions with a zero numerator and/or denominator. These are $\frac{n}{0}$, $\frac{0}{0}$, and $\frac{0}{d}$. He defined $\frac{0}{0} = 0$, but was silent on the properties of the other two classes of fractions. However, if we accept $r = \frac{r}{1}$, which was prefigured before about 890 CE and known in 1150 CE, then it is a corollary of Brahmegupta's arithmetic that $\frac{0}{d} = 0$, for all non-zero real d. This leaves the semantics of the class $\frac{n}{0}$ unspecified for all non-zero real n. Bhascara, 1150 CE, addresses these semantics.

2.3 Bhascara's 1150 CE Texts

Bhascara updated Brahmegupta's texts in a total of 502 verses. Two matters are of interest to us, the handling of infinitesimals and infinities.

The arithmetic of infinities and the fraction $\frac{0}{0}$ depends upon whether we use Addition A or Addition B, described in Section 2.1.2. Bhascara and his commentators used Addition B. Bhascara says the following.

L36 *Rule for addition and subtraction of fractions:*[2] *half a stanza. The sum or [in the case of subtraction] the difference of fractions having a common denominator, is [taken]. Unity*[1] *is put denominator of a quantity*[2] *which has no divisor.*[3]

This is consistent with Addition B, in as much as a common denominator is assumed, prior to addition and subtraction, and is not enforced during the calculation of addition and subtraction. That Bhascara follows Addition B is asserted by his commentators in the illustrative calculations below.

The footnotes, which are renumbered across a page boundary, are of no interest to us.

We translate Verse L36 as follows.

L36 {*L36.1*} *Rule for addition and subtraction of fracterms.* {*L36.2*} *The sum or in the case of subtraction the difference of fracterms having a common denominator is taken.* {*L36.3*} *Unity,* 1*, is put denominator of a number,* r*, which has no divisor. Thus,* $r = \frac{r}{1}$.

Rehearsals: (G2.2, L36.2), (G2.3, L36.2).

Novelties: L36.3.

Note that L36.3 is a statement of our Definition 1.6 that establishes division as an operation that introduces a fraction.

2.3.1 Infinitesimals

Bhascara, 1150 CE, introduces infinitesimal numbers in Verse L44. This verse also rehearses Brahmegupta's sign conventions for zero, except that Bhascara does not explicitly recognise Brahmegupta's definition $\frac{0}{0} = 0$.

L44 *Rule for arithmetical process relative to cipher: two couplets. In addition, cipher makes the sum equal to the additive.*[2] *In involution and [evolution]*[3] *the result is cipher. A definite quantity,*[4] *divided by cipher, is the submultiple of nought.*[5] *The product of cipher is nought: but it must be retained as a multiple of cipher,*[6] *if any further operation impend. Cipher having become a multiplier, should nought afterwards become a divisor, the definite quantity must be understood to be unchanged. So likewise any quantity, to which cipher is added, or from which it is subtracted, [is unaltered.]*

Most of the footnotes are of no interest to us. The fifth footnote is interesting, but is better handled in the Vija-ganita, as discussed in the next section on Infinities. The sixth footnote tells us how to construct an infinitesimal as a tuple.

In the whole of Colebrooke's text, *involution* is raising to a square or cubic power and *evolution* is extracting a square or cube root.

In the sentence, *A definite quantity,*[4] *divided by cipher, is the submultiple of nought,*[5] we understand that the *definite quantity* is a real number, r. When r is divided by cipher, $x = r \div 0$, we obtain the vulgar fraction $x = \frac{r}{0}$, without remainder, because 0 is a submultiple, or exact divisor, of r.

Now we come to infinitesimals. The sixth Footnote says: [6] *Chaguna, a quantity which has cipher for its multiplier. Cipher is set down by the side of the multiplicand, to denote it.* Thus, we write $0 \times r = \langle 0, r \rangle$, where the tuple, $\langle 0, r \rangle$ records the multiplicand, r, "beside" 0.

We translate Verse L44 as follows. The word "impending" is still current in English, it means "arising later." Similarly, "impend" means "arises later."

L44 {L44.1} Rule for arithmetical processes relative to zero. {L44.2} In addition, zero makes the sum equal to the additive, as $r + 0 = r$. {L44.3} The square, cube, square root, and cube root of zero are all zero. {L44.4} A definite number, r, divided by zero, is a fraction with zero as the denominator: $r \div 0 = \frac{r}{0}$. {L44.5} The product of zero is zero: but the multiplicand must be retained as a multiple of zero in the tuple $0 \times r = \langle 0, r \rangle$, in case any further operation impend. {L44.6} Zero having become a multiplier, should zero afterwards become a divisor, the definite number, r, must be understood to be unchanged, as $\langle 0, r \rangle \div 0 = r$. {L44.7} So likewise any number, to which zero is added, or from which it is subtracted, is unaltered, as $r + 0 = r$ and $r - 0 = r$.

Rehearsals: (C31.7, L44.2), (C35.3, L44.4), (C35.5, L44.4), (C34.4, L44.5), (C34.5, L44.5), (C31.6, L44.7), (C31.7, L44.7), (C31.8, L44.7).

Novelties: L44.5, L44.6.

Note that L44.3 is not a novelty. It deals with operations that Brahmegupta describes, but which we have not indexed because they do not deal directly with the four fundamental operations of arithmetic: addition, subtraction, multiplication, and division.

Note that L44.4 is a particular case of our more general Definition 1.6 that establishes division as an operation that introduces a fraction.

Note that at L44.5 and L44.6, Bhascara introduces infinitesimal numbers as tuples $\langle 0, r \rangle$.

Verses V128 - V152 deal with solutions to quadratic equations. Two of these, V135 and V136, give examples of infinitesimal arithmetic, which also advance our understanding of Bhascara's infinities.

V135 *Example: What number being divided by cipher, and having the original quantity added to the quotient and nine subtracted from this sum,*[2] *and the consequent remainder being squared and having its square-root added to that square, and the whole being then multiplied by cipher, will amount to ninety?*[3]

Here the number is put ya 1*. This divided by cipher, is ya* $\frac{1}{0}$*. (The addition and subtraction being made, it is still ya* $\frac{1}{0}$*. This squared is ya v* $\frac{1}{0}$*. The root added to it makes ya v* $\frac{1}{0}$ *ya* $\frac{1}{0}$*. Multiplied by cipher, the multiplier and divisor being alike vanish, leaving ya v* 1 *ya* 1*.) Hence multiplying [the equation] by four and adding one, and proceeding as before, the number is found 9.*

Here the statement of the problem is clear, without the footnotes, which do not advance our understanding. The solution is stated to be 9. The solution method, beginning *Here the number is put ya* 1 is partially opaque to us so we attempt no translation below. Instead we insert modern algebra into the statement of the problem, but repeat the solution verbatim. It might help the reader to know that "*ya*" introduces a variable and the letter "*v*" means squaring, that is raising to the second power. As usual, in Sanscrit, the infix operation of addition was not written!

V135 {V135.1} Example: What number, x, being divided by zero, as $x \div 0 = \frac{x}{0}$, and having the original quantity added to the quotient, as $\frac{x}{0} + x$, and nine subtracted from this sum, as $\frac{x}{0} + x - 9$, and the result being squared, as $(\frac{x}{0} + x - 9)^2$, and having its square-root added to that square, as $(\frac{x}{0} + x - 9)^2 + (\frac{x}{0} + x - 9)$, and the whole being then multiplied by zero, as $((\frac{x}{0} + x - 9)^2 + (\frac{x}{0} + x - 9)) \times 0$, will amount to ninety, as $((\frac{x}{0} + x - 9)^2 + (\frac{x}{0} + x - 9)) \times 0 = 90$?

{V135.2} Here the number is put *ya* 1. {V135.3} This divided by zero, is *ya* $\frac{1}{0}$. {V135.4} (The addition and subtraction being made, it is still *ya* $\frac{1}{0}$. {V135.5} This squared is *ya v* $\frac{1}{0}$. {V135.6} The root added to it makes *ya v* $\frac{1}{0}$ *ya* $\frac{1}{0}$. {V135.7} Multiplied by zero, the multiplier and divisor being alike vanish, leaving *ya v* 1 *ya* 1.) {V135.8} Hence multiplying the equation by four and adding one, and proceeding as before, the number is found 9.

We state the equation as follows.

$$((\frac{x}{0} + x - 9)^2 + (\frac{x}{0} + x - 9)) \times 0 = 90 \tag{1}$$

The left hand side is a multiplication by zero which produces the result zero, according to Brahmegupta in C34.4 and C34.5. See Subsection 2.2.4. However, this would produce the contradiction $0 = 90$, which is clearly not intended. But a contradiction need not arise, if the zero used here is the infinitesimal, introduced by Bhascara in Verse L44, discussed above.

Now, as a preliminary step:

$$\frac{x}{0} + x = \frac{x}{0} + \frac{x}{1} = \frac{x \times 1 + x \times 0}{0 \times 1} = \frac{x}{0} \tag{2}$$

Similarly $\frac{x}{0} - 9 = \frac{x}{0}$. This agrees with V135.4. Thus Equation 1 simplifies to:

$$((\frac{x}{0})^2 + \frac{x}{0}) \times 0 = (\frac{x^2}{0} + \frac{x}{0}) \times 0 = 90 \quad (3)$$

At this point we have two ways forward. According to Addition A, in Section 2.1.2, we have:

$$(\frac{x^2}{0} + \frac{x}{0}) \times 0 = (\frac{x^2 \times 0 + x \times 0}{0 \times 0}) \times 0 = \frac{0}{0} \times 0 = 0 \times 0 = 0 = 90 \quad (4)$$

But this is the contradiction $0 = 90$. However, using Addition B, we have:

$$(\frac{x^2}{0} + \frac{x}{0}) \times 0 = \frac{x^2 + x}{0} \times 0 = 90 \quad (5)$$

Cancelling the infinitesimal zero in the numerator and denominator gives:

$$x^2 + x = 90 \quad (6)$$

This agrees with V135.7 so Bhascara follows Addition B and allows the cancellation of a common factor of zero. Alternatively, using Bhascara's infinitesimal tuples in the numerator, we have the same result:

$$\frac{x^2 + x}{0} \times 0 = \frac{x^2 + x}{0} \times \frac{0}{1} = \frac{(x^2 + x) \times 0}{0 \times 1} = \frac{\langle 0, x^2 + x \rangle}{0} = x^2 + x = 90 \quad (7)$$

Our derivation suggests that Bhascara was using his tuples. But note an elision. In the denominator we have written $0 \times 1 = 0$, but should we have written $0 \times 1 = \langle 0, 1 \rangle$, leading to $\langle 0, x^2 + x \rangle / \langle 0, 1 \rangle = x^2 + x$? Our uncertainty reflects Bhascara's underspecification of infinitesimal arithmetic.

Proceeding from Equation 7, rearranging gives a quadratic equation in the modern canonical form:

$$x^2 + x - 90 = 0 \quad (8)$$

This has solutions $x_1 = 9$ and $x_2 = -10$. The positive root agrees with Bhascara's solution, 9, at V135.8, confirming that Bhascara follows Addition B in Section 2.1.2 and allows the cancellation of a common factor of zero.

Notice that at V135.7, Bhascara says *the multiplier and divisor being alike vanish*, in other words $\frac{r}{r} = 1$, but this contradicts Brahmegupta's $\frac{0}{0} = 0$ at C35.3 in Subsection 2.2.5. Bhascara does not rehearse C35.3 so we suppose that he does not accept Brahmegupta's definition $\frac{0}{0} = 0$.

Bhascara gives a second example of an infinitesimal calculation.

V136 *Example: Say what is the number, which having its half added to it, and being multiplied by cipher, and the product squared, and added to twice the root of that square, and this sum being divided by cipher, becomes fifteen?*

The number is put ya 1. *This, having its half added to it, becomes ya* $\frac{3}{2}$. *Being multiplied by cipher, it is not to be made nought but* *to be considered as multiple of cipher, further operations impending. Wherefore ya* $\frac{3}{2} \cdot 0$*, being squared, and having twice the root added, becomes ya v* $\frac{9}{4} \cdot 0$ *ya* $\frac{12}{4} \cdot 0$*. This is divided by cipher: and here, as before, the multiplier being* 0*, and the divisor* 0*, both multiplicator and divisor, as being equal, vanish; and the quantity is unaltered:*[1] *viz. ya v* $\frac{9}{4}$ *ya* $\frac{12}{4}$*. Equating with fifteen, reducing to a common denomination, and then dropping the denominator, the two sides of the equation by preparation become*

ya v 9 *ya* 12 *ru* 0
ya v 0 *ya* 0 *ru* 60

Adding four, and extracting the square-roots, the value of yavat-tavat by equal subtraction comes out 2.

It would appear that Colebrooke has used a dot to indicate multiplication, as in $a \cdot b = a \times b$, where Sanscrit authors would be expected to use a space, as with all infix operators. Especially as a dot usually means negation.

The footnote does not add to our understanding. We now translate the verse as follows, again leaving the solution verbatim.

V136 {V136.1} Example: Say what is the number, x, which having its half added to it, as $x + \frac{x}{2} = \frac{3x}{2}$, and being multiplied by zero, $\frac{3x}{2} \times 0$, and the product squared, $(\frac{3x}{2} \times 0)^2$, and added to twice the root of that square, $(\frac{3x}{2} \times 0)^2 + 2(\frac{3x}{2} \times 0)$, and this sum being divided by zero, $((\frac{3x}{2} \times 0)^2 + 2(\frac{3x}{2} \times 0))/0$, becomes fifteen, $((\frac{3x}{2} \times 0)^2 + 2(\frac{3x}{2} \times 0))/0 = 15$?

{V136.2} The number is put *ya* 1. {V136.3} This, having its half added to it, becomes *ya* $\frac{3}{2}$. {V136.4} Being multiplied by zero, it is not to be made zero but is to be considered a multiple of zero, further operations impending. {V136.5} Wherefore *ya* $\frac{3}{2} \cdot 0$, being squared, and having twice the root added, becomes *ya v* $\frac{9}{4} \cdot 0$ *ya* $\frac{12}{4} \cdot 0$. {V136.6} This is divided by zero: and here, as before, the multiplier being 0, and the divisor 0, both multiplicator and divisor, as being equal, vanish; and the quantity is unaltered:[1] viz. *ya v* $\frac{9}{4}$ *ya* $\frac{12}{4}$. {V136.7} Equating with fifteen, reducing to a common denominator, and then dropping the denominator, the two sides of the equation by preparation become:

ya v 9 *ya* 12 *ru* 0
ya v 0 *ya* 0 *ru* 60

{V136.8} Adding four, and extracting the square-roots, the value of yavat-tavat by equal subtraction comes out 2.

We state the equation as follows.

$$((\frac{3x}{2} \times 0)^2 + 2(\frac{3x}{2} \times 0))/0 = 15 \qquad (9)$$

Expanding:

$$((\frac{9x^2}{4} \times 0) + (\frac{6x}{2} \times 0))/0 = 15 \tag{10}$$

Rewriting with the common denominator, 4, on the left-hand side gives:

$$((\frac{9x^2}{4} \times 0) + (\frac{12x}{4} \times 0))/0 = 15 \tag{11}$$

Multiplying out by the common denominator of the left hand side, 4, gives:

$$((9x^2 \times 0) + (12x \times 0))/0 = 60 \tag{12}$$

Cancelling the infinitesimal zeros gives:

$$9x^2 + 12x = 60 \tag{13}$$

This agrees with V136.7 where the first row of the tableau shows the left hand side, *ya v* 9 *ya* 12 *ru* $0 = 9x^2+12x+0$, and the second row shows the right hand side, *ya v* 0 *ya* 0 *ru* $60 = 0x^2+0x+60$. Therefore, we are confident that we have faithfully reproduced Bhascara's calculation.

Rearranging gives a quadratic equation in the modern canonical form:

$$9x^2 + 12x - 60 = 0 \tag{14}$$

This has solutions $x_1 = 2$ and $x_2 = \frac{-10}{3}$. The positive root agrees with Bhascara's solution, 2, at V136.8, so we are again confident that we have faithfully reproduced Bhascara's calculation.

Now, let us rework Bhascara's calculation using his tuples. We begin by rehearsing Equation 9:

$$((\frac{3x}{2} \times 0)^2 + 2(\frac{3x}{2} \times 0))/0 = 15 \tag{15}$$

$$(\langle 0, \frac{3x}{2} \rangle^2 + 2\langle 0, \frac{3x}{2} \rangle)/0 = 15 \tag{16}$$

$$(\langle 0, \frac{9x^2}{4} \rangle + 2\langle 0, \frac{6x}{2} \rangle)/0 = 15 \tag{17}$$

$$(\langle 0, \frac{9x^2}{4} \rangle + \langle 0, \frac{12x}{4} \rangle)/0 = 15 \tag{18}$$

$$(\langle 0, 9x^2 \rangle + \langle 0, 12x \rangle)/0 = 60 \tag{19}$$

$$9x^2 + 12x = 60 \tag{20}$$

This is the same as Equation 17 so we see that Bhascara's tuples are effective in solving this quadratic equation involving division by infinitesimal zeros.

Notice that in V136.5, Bhascara writes his infinitesimal tuples with zero in the right-hand place, as $\frac{3}{2} \cdot 0$, but we write it in the left-hand place, $\langle 0, d \rangle$. This facilitates the identification $\langle n, d \rangle = \frac{n}{d}$, with $n = 0$, such that $\langle 0, d \rangle = \frac{0}{d}$, whence we could implement Bhascara's infinitesimal algebra in terms of fracterms $\frac{0}{d}$. But Bhascara's $0/0 = 1$ contradicts Brahmegupta's $0/0 = 0$, which contradiction must be handled by some means. One way is to introduce infinitesimals as a different category of objects, tuples, as against fracterms. A second way is to say that as Bhascara nowhere explicitly rehearses Brahmegupta's definition $\frac{0}{0} = 0$, he does not accept it. But this second way fails, because it is a theorem of Bhascara's arithmetic that $\frac{0}{0} = \frac{0\times 0}{0} = \frac{\langle 0,0\rangle}{0} = 0$. Whence Bhascara's infinitesimal arithmetic is irredeemably inconsistent, because it enforces $\frac{0}{0} = 0$.

In summary, Bhascara introduced an inchoate arithmetic of infinitesimal tuples, following Addition B for the addition and subtraction of fracterms. We now define the fragment of infinitesimal arithmetic that Bhascara explicitly described.

L44.8 Definition: infinitesimal tuples, $\langle 0, r\rangle$, arise from the product of any real number, r, with zero, as:

$$r \times 0 = 0 \times r = \langle 0, r\rangle$$

L44.9 Definition: for all real numbers, r, infinitesimal tuples, $\langle 0, r\rangle$, are divided by zero, as:

$$\frac{\langle 0, r\rangle}{0} = r$$

2.3.2 Infinities

Brahmegupta supplied a class of fractions, $\frac{n}{0}$ for all non-zero n, which Bhascara identifies as infinities. In this section, we set out the mathematical context of this identification. In a later section, we derive Brahmegupta's and Bhascara's related arithmetics.

Bhascara rehearses much of Brahmegupta's arithmetic, but does not explicitly acknowledge Brahmegupta's definition $\frac{0}{0} = 0$. Indeed, Bhascara's infinitesimal arithmetic, presented above, contradicts this definition.

The footnotes, which are sometimes renumbered across page boundaries in the following verses, do not advance our understanding.

L29 *Rule. The numerator and denominator*[4] *being multiplied reciprocally by the denominators of the two quantities,*[5] *they are thus reduced to the same denomination. Or both numerator and denominator may be multiplied by the intelligent calculator into the reciprocal denominators abridged by a common measure.*

We translate Verse L29 as follows.

L29 {L29.1} Rule to transform two fracterms so that they have a common denominator. {L29.2} The numerators, n_1 and n_2, and denominators, d_1 and d_2, being multiplied by the opposite denominators of the two fracterms, $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$, giving $\frac{n_1\times d_2}{d_1\times d_2}$ and $\frac{n_2\times d_1}{d_2\times d_1}$, they are thus reduced to the same denominator, $d_1 \times d_2 = d_2 \times d_1$. {L29.2} Or both numerator and denominator may be multiplied by the intelligent calculator into the opposite denominators reduced by a common factor. Thus, when $d_1 = d_3 \times k$ and $d_2 = d_4 \times k$, the common factor, k, is cancelled, giving $\frac{n_1\times d_4}{d_1\times d_4}$ and $\frac{n_2\times d_3}{d_2\times d_3}$.

Rehearsals: (G2.1, L29.2).

Note that in L29.2 Bhascara separates reduction to a common denominator from addition and subtraction, which is independent of Addition A and Addition B.

L36 *is presented above in Section 2.3. It shows that Bhascara used Addition B.*

L38 *Rule for multiplication of fractions:*[4] *half a stanza. The product of the numerators, divided by the product of denominators, [gives a quotient, which] is the result of multiplication of fractions.*

We translate Verse L38 as follows.

L38 {L38.1} Rule for multiplication of fracterms $\frac{n_i}{d_i}$. {L38.2} The product of the numerators, $n_1 \times n_2 \times \ldots n_i$, divided by the product of denominators, $d_1 \times d_2 \times \ldots d_i$, gives a quotient, $\frac{n_1 \times n_2 \times \ldots n_i}{d_1 \times d_2 \times \ldots d_i}$ which is the result of multiplication of fracterms $\frac{n_1}{d_1} \times \frac{n_2}{d_2} \times \ldots \frac{n_i}{d_i}$.

Rehearsals: (G3.2, L38.2).

L40 *Rule for division of fractions:*[5] *half a stanza. After reversing the numerator and denominator of the divisor, the remaining process for division of fractions is that of multiplication.*

We translate Verse L40 as follows.

L40 {L40.1} Rule for division of fracterms. {L40.2} After reversing the numerator and denominator of the divisor, the remaining process for division of fracterms is that of multiplication. Thus, $\frac{n_1}{d_1} \div \frac{n_2}{d_2} = \frac{n_1}{d_1} \times \frac{d_2}{n_2}$.

Rehearsals: (G4.2, L40.2).

Note that Bhascara's L40.2 rewrites division as multiplication by the reciprocal. It encompasses our lexical Definition 1.7 and 1.8.

L44 The arithmetics of zero as an algebraic number and as an infinitesimal number are set out in Verse L44 in Subsection 2.3.1 above.

We now review Bhascara's understanding of positive and negative.

V3 *Rule for addition of affirmative and negative quantities: half a stanza. In the addition of two negative or two affirmative*[1] *quantities, the sum must be taken: but the difference of an affirmative and a negative quantity is their addition.*[2]

The footnotes do not advance our understanding, but Verse V3 justifies rewriting a subtraction as a sum with the negated subtrahend, as $x - y = x + (-y)$.

V3 {V3.1} Rule for addition of positive and negative numbers. {V3.2} In the addition of two negative, $-p_1$ and $-p_2$, or two positive, p_1 and p_2, numbers, the sum must be taken, as $p_1 + p_2 = p_3$ and $(-p_1) + (-p_2) = -(p_1 + p_2) = -p_3$: but the difference of a positive and a negative number is their addition: $p_1 - (-p_2) = p_1 + p_2$.

Rehearsals: (C31.2, V3.2), (C31.3, V3.2), (C31.4, V3.2), (C31.5, V3.2).

V5 *Rule for subtraction of positive and negative quantities: half a stanza. The quantity to be subtracted being affirmative, becomes negative; or, being negative, becomes affirmative: and the addition of the quantities is then made as above directed.*[3]

This reasserts $x - y = x + (-y)$. Now Colebrooke's Footnote, which quotes the commentator Crishna, commenting on Crishna Bhatta's comment is critical.

Footnote: [3] *In demonstrating this rule, the commentator CRISHNA BHATTA observes, that 'here negation is of three sorts, according to place, time, and things. It is, in short, contrariety. Wherefore the Lilavati, §166, expresses "The segment is negative, that is to say, is in the contrary direction." As the west is the contrary of east; and the south the converse of north. Thus, of two countries, east and west, if one be taken as positive, the other is relatively negative. So when motion to the east is assumed to be positive, if a planet's motion be westward, then the number of degrees equivalent to the planet's motion is negative. In like manner, if a revolution westward be affirmative, so much as a planet moves eastward, is in respect of a western revolution negative. The same may be understood in regard to south and north, &c. That prior and subsequent times are relatively to each other negative, is familiarly understood in reckoning of days. So in respect of chattels, that, to which a man bears the relation of owner, is considered as positive in regard to him: and the converse [or negative quantity] is that to which another person has the relation of owner. Hence so much as belongs to Yajnyadatta in the wealth possessed by Devadatta, is negative in respect of Devadatta. The commentator gives as an example the situation of Pattana (Patna) and Prayaga (Allahabad) relatively to Anandatana (Benares). Pattana on the Ganges bears east of Varanasi, distant fifteen yojanas; and Prayaga on the confluence of the Ganga and Yamuna, bears west of the same, distant eight yojanas. The interval or difference is twenty-three yojanas; and is not obtained but by addition of the numbers. Therefore, if the difference between two contrary quantities be required, their sum must be taken.*

The reference to Verse L166 does not advance our understanding. Here "&c" means "etcetera."

Crishna Bhatta, confirmed by Crishna, describes negative as the opposite of positive and gives the specific example of the west being the opposite of the east. This east-west relationship is critical to our understanding of Bhascara's unsigned infinities.

A footnote, by Crishna, to Verse V13, says: *Cipher is neither positive nor negative: and it is therefore exhibited with no distinction of sign. No difference arises from the reversing of it; and none is here shown.* Thus, we understand that in Sanscrit arithmetic, zero is unsigned. The question then arises, are infinities, being fractions with a zero denominator, signed or unsigned?

We translate Verse V5 as follows.

V5 {V5.1} Rule for subtraction of positive and negative numbers from an arbitrary number r. {V5.2} The number to be subtracted being positive, p in $r - p$, becomes negative, as $(-p)$; or, being negative, $(-p)$ becomes positive, p: and the addition of the numbers is then made as above directed. Thus, $r - p = r + (-p)$ and $r - (-p) = r + p$.

Rehearsals: (C31.4, V5.2), (C32.4, V5.2), (C32.6, V5.2)

V7 *Rule for multiplication [and division] of positive and negative quantities: half a stanza. The product of two quantities both affirmative, is positive.*[1] *When a positive quantity and a negative one are multiplied together, the product is negative.*[2] *The same is the case in division.*

We translate Verse V7 as follows.

V7 {V7.1} Rule for multiplication and division of positive and negative numbers. {V7.2} The product of two numbers both positive, is positive. When a positive number and a negative number are multiplied together, the product is negative. The same is the case in division.

Rehearsals: (C34.2, V7.2), (C35.2, V7.2), (C35.4, V7.2)

Notice that in Verse V7, Bhascara does not say how to multiply or divide two negative numbers. This case is given in Verse V8.

V8 *Example: half a stanza (completing §6). What is the product of two multiplied by three, positive by positive; and negative by negative; or positive by negative?*

Statement: 2. 3. *Affirmative multiplied by affirmative is affirmative. Product* 6.

Statement: $\dot{2}$. $\dot{3}$. *Negative multiplied by negative is positive. Product* 6.

Statement: 2. $\dot{3}$ *[or* $\dot{2}$. 3.*] Positive multiplied by negative [or, negative by positive] is negative. Product* $\dot{6}$.

The result is the same, if the multiplicator be multiplied by the multiplicand.[1]

DIVISION.
Rule. The same is the case in division (§7).[2]

In the first footnote, Crishna says: [1] *It is thus intimated, that either quantity may at pleasure be treated as multiplicator, and the other as multiplicand: and conversely.* This says that multiplication is commutative.

In the second footnote, Crishna says: [2] *If both the dividend and the divisor be affirmative, or both negative, the quotient is affirmative: but, if one be positive and the other negative, the quotient is negative.* We translate this as "If both the dividend and the divisor be positive, or both negative, the quotient is positive: but, if one be positive and the other negative, the quotient is negative."

Colebrooke has confused the notation by writing a full stop after the numbers "2." and "3." We translate Verse V8 as follows.

V8 {V8.1} Example: half a stanza (completing §6). What is the product of two multiplied by three, positive by positive; and negative by negative; or positive by negative?

{V8.2} Statement: $2 \times 3 = 6$. Positive multiplied by positive is positive.

{V8.3} Statement: $(-2) \times (-3) = 6$. Negative multiplied by negative is positive.

{V8.4} Statement: $2 \times (-3) = -6$ or $(-2) \times 3 = -6$. Positive multiplied by negative or, negative by positive, is negative.

{V8.5} The result is the same, if the multiplicator be multiplied by the multiplicand.

{V8.6} DIVISION.
Rule. The same is the case in division, see Verse V7.

Rehearsals: (C34.3, V8.2), (C34.3, V8.3), (C34.2, V8.4), (C35.2, V8.2), (C35.2, V8.3), (C35.4, V8.4)

V12 *Rule for addition and subtraction of cipher: part of a stanza. In the addition of cipher, or subtraction of it, the quantity,*[1] *positive or negative, remains the same. But, subtracted from cipher, it is reversed.*[2]

The footnotes do not advance our understanding. We translate Verse V12 as follows.

V12 {V12.1} Rule for addition and subtraction of zero. {V12.2} In the addition of zero, or subtraction of it, a positive number, p, or a negative number, $-p$, remains the same, as $\pm p + 0 = \pm p$ and $\pm p - 0 = \pm p$. {V12.3} But, subtracted from zero, it is reversed, as $0 - (\pm p) = \mp p$.

Rehearsals: (C31.6, V12.2), (C31.7, V12.2), (C32.4, V12.3).

V14 *Rule: (completing the stanza, § 12.) In the multiplication and the rest of the operations*[1] *of cipher, the product is cipher; and so it is in multiplication by cipher: but a quantity, divided by cipher, becomes a fraction the denominator of which is cipher.*[2]

The first footnote says that *the rest of the operations of cipher* are the operations of division, square, and square root.

The second footnote, by Crishna, reads: [2] *The more the multiplicand is diminished, the smaller is the product; and, if it be reduced in the utmost degree, the product is so likewise: now the utmost diminution of a quantity is the same with the reduction of it to nothing: therefore, if the multiplicand be nought, the product is cipher. In like manner, as the multiplier decreases, so does the product; and, if the multiplier be nought, the product is so too. In fact multiplication is repetition: and, if there be nothing to be repeated, what should the multiplicator repeat, however great it be?*

So, if the dividend be diminished, the quotient is reduced: and, if the dividend be reduced to nought, the quotient becomes cipher.

As much as the divisor is diminished, so much is the quotient raised. If the divisor be reduced to the utmost, the quotient is to the utmost increased. But, if it can be specified, that the amount of the quotient is so much, it has not been raised to the utmost: for a quantity greater than that can be assigned. The quotient therefore is indefinitely great, and is rightly termed infinite.

Thus, Crishna employs limits to argue that multiplication by zero gives the product zero and division of a non-zero number, n, by zero, expressed in the fraction $\frac{n}{0}$, gives a quotient that is unboundedly great.

It was the custom of Sanscrit mathematicians to calculate magnitudes and then to apply signs. Crishna describes the calculation of an infinite magnitude, but gives no indication of whether he thinks infinite fractions are signed or unsigned.

We translate Verse V14 as follows.

V14 {V14.1} Rule: (completing the stanza, § 12.) The result of the following operations on zero is zero: {V14.2} multiplication, $r \times 0 = 0 \times r = 0$; {V14.3} division, $0 \div r = 0$; {V14.4} square, $0^2 = 0$; and {V14.5} square root, $\sqrt{0} = 0$. {V14.6} But a number, r, divided by zero, becomes a fraction the denominator of which is zero, $\frac{r}{0}$.

Rehearsals: (C34.4, V14.2), (C34.5 V14.2), (C35.3, V14.2), (C35.6, V14.2), (C35.5, V14.6)

Notice that V14.3 implies $\frac{0}{0} = 0$, but this is our interpretation of Crishna's interpretation of Bhascara's text. It is not explicit in Bhascara's text. However, V14.6 is part of Bhascara's text and it implies, but does not explicitly state, the existence of $\frac{0}{0}$.

We summarise this by saying that the equality $\frac{0}{0} = 0$ is implicit but not explicit in Bhascara's texts. However, while the existence of $\frac{0}{0}$ is consistent with Bhascara's account of infinitesimals, in Subsection 2.3.1 above, the algebraic equality $\frac{0}{0} = 0$ generally contradicts Bhascara's infinitesimal equality $\frac{\langle 0,r\rangle}{0} = r$. This is a complication we must deal with when giving an account of his arithmetics. As previously stated, we handle this by saying that Bhascara revokes Brahmegupta's definition $\frac{0}{0} = 0$ in the arithmetic of fracterms and introduces infinitesimal tuples as a distinct kind of object. If we were to treat Bhascara's tuples $\langle 0, r\rangle$ as fracterms $\frac{0}{r}$, then we would have to take some action to handle this contradiction.

Verse V15 presents the fraction $\frac{3}{0}$ and a comment by Crishna, reported in Colebrooke's fifth footnote, generalises this to $\frac{n}{0}$ for all non-zero numbers n.

V15 *Example: half a stanza. Tell me the product of cipher multiplied by two;*[3] *and the quotient of it divided by three, and of three divided by cipher; and the square of nought; and its root.*

Statement : Multiplicator 2. *Multiplicand* 0. *Product* 0.

[Statement : Multiplicator 0. *Multiplicand* 2. *Product* 0.[4]*]*

Statement : Dividend 0. *Divisor* 3. *Quotient* 0.

Statement : Dividend 3. *Divisor* 0. *Quotient the fraction* $\frac{3}{0}$.

This fraction, of which the denominator is cipher, is termed an infinite quantity.[5]

The last sentence is satisfactory when discussing the particular fraction $\frac{3}{0}$: *This fraction, of which the denominator is cipher, is termed an infinite quantity.*

But this interpretation runs into trouble when it is generalised, below, to all real r divided by zero, in the fraction $\frac{r}{0}$, because it implies that $\frac{0}{0}$ is infinite in the sense of having an unbounded magnitude, despite being a product with zero: $0 \times \frac{r}{0} = \frac{0}{1} \times \frac{r}{0} = \frac{0 \times r}{1 \times 0} = \frac{0}{0}$.

The third and fourth footnotes do not advance our understanding, but the fifth footnote, by Crishna, is critical. We present it in two parts. The first part is an argument from limits, the second is an argument from geometry.

The first part reads: [5] *Ananta-rasi, infinite quantity. Cha-hara, fraction having cipher for its denominator. This fraction, indicating an infinite quantity, is unaltered by addition or subtraction of a finite quantity. For, in reducing the quantities to a common denominator, both the numerator and denominator of the finite quantity, being multiplied by cipher, become nought: and a quantity is unaltered by the addition or subtraction of nought. The numerator of the infinite fraction may indeed be varied by the addition or subtraction of a finite quantity, and so it may by that of another infinite fraction: but whether the finite numerator of a fraction, whose denominator is cipher, be more or less, the quotient of its division by cipher is alike infinite.*

Here we must take care not to assert that Crishna identifies any fraction with a zero denominator as infinite, instead we read his *finite* numerator as excluding zero. If this were not the case, we would run into two problems. The first problem is that Brahmegupta defines $\frac{0}{0} = 0$, at C35.3 in our Subsection 2.2.5, but, in the modern reading, 0 is a finite quantity, not an infinite one, which would contradict Crishna. The second problem is that Bhascara's infinitesimal tuples can also give a finite result, which would also contradict Crishna:

$$\frac{0}{0} = \frac{0 \times 1}{0} = \frac{\langle 0, 1 \rangle}{0} = 1$$

The most parsimonious solution to these problems is to accept that Bhascara and his commentator, Crishna, do not regard the fraction $\frac{0}{0}$ as being infinite, whence we assert that their accounts of infinities do not apply to this corner case.

Even so, Crishna's claim that an infinite quantity is unchanged by the addition of a finite quantity is questionable. If we follow Addition A of Brahmegupta's G2.2A, which in any case executes the reduction to a common denominator required as preprocessing in Addition B, G2.2B, then, for all real non-zero n_i and d_2, we have:

$$\frac{n_1}{0} + \frac{n_2}{d_2} = \frac{n_1 \times d_2 + n_2 \times 0}{0 \times d_2} = \frac{n_1 \times d_2}{0} \qquad (21)$$

Thus, we see that the infinite fraction $\frac{n_1}{0}$ has, contrary to Crishna, been transformed to the infinite fraction $\frac{n_1 \times d_2}{0}$. We can save Crishna, if we suppose he accepted $\frac{0}{0} = 0$ and intended a calculation such as:

$$\frac{n_1}{0} + \frac{n_2}{d_2} = \frac{n_1}{0} + \frac{n_2 \times 0}{d_2 \times 0} = \frac{n_1}{0} + \frac{0}{0} = \frac{n_1}{0} + 0 = \frac{n_1}{0}$$

Then the remainder of the comment, but now excluding $\frac{0}{0} = 0$, is unproblematic. Where Crishna says, *. . . whether the finite numerator of a fraction, whose denominator is cipher, be more or less, the quotient of its division by cipher is alike infinite*, we again take it that zero was not considered a finite number, whence we read *alike infinite* as saying that the class of fractions with a non-zero numerator, n, and zero denominator has infinite or unbounded magnitude, $|\frac{n}{0}|$; but are the infinite fractions themselves, $\frac{n}{0}$, signed or unsigned? This question is refined in the second part of the comment that Colebrooke credits to Crishna.

This is illustrated by the same commentator through the instance of the shadow of a gnomon, which at sunrise and sunset is infinite; and is equally so, whatever height be given to the gnomon, and whatever number be taken for radius, though the expression will be varied. Thus, if radius be put 120; and the gnomon be 1, 2, 3, or 4; the expression deduced from the proportion, as sine of sun's altitude is to sine of zenith distance; so is gnomon to shadow; becomes $\frac{120}{0}$, $\frac{240}{0}$, $\frac{360}{0}$ *or* $\frac{480}{0}$. *Or, if the gnomon be, as it is usually framed, 12 fingers, and radius be taken at 3438, 120, 100, or 90 ; the expression will be* $\frac{41256}{0}$, $\frac{1440}{0}$, $\frac{1200}{0}$ *or* $\frac{1080}{0}$; *which are all alike infinite.*

Notice that if the gnomon has a height of zero fingers then at both sunrise and sunset, the shadow has length $\frac{0}{0}$; but a gnomon of zero height casts no shadow or, put another way, it casts a shadow of zero length, so, here, $\frac{0}{0} = 0$, rehearsing Brahmegupta. This geometrical argument leads to the conclusion that $\frac{0}{0}$ is not infinite – because it is zero!

Crishna says that all of the above positive numerators, divided by zero, are alike infinite. That is, all of the fracterms $\frac{p_i}{0}$ have an unboundedly large magnitude, $|\frac{p_i}{0}|$; but this does not necessarily mean that all of these fracterms form an equivalence class $\frac{p_i}{0} = \infty$. In fact they do not, because some sums and differences, $\frac{p_i}{0} - \frac{p_i}{0} = \frac{0}{0}$, using Addition A or Addition B, and the magnitude $|\frac{0}{0}|$ is not unboundedly large, because it is a product of zero or because it is zero by the above geometrical argument. Thus, Crishna and, we suppose, Bhascara, described a multitude of infinite fractions with any non-zero numerator, n_i, such that $\frac{n_i}{0} = \infty_i$, not a single infinite number $\frac{n_i}{0} = \infty$.

The question now arises as to whether Bhascara's infinite fractions, $\frac{n_i}{0}$, are signed or unsigned. In the example of the gnomon, the infinite shadows occur at sunrise (with the sun in the east, casting the shadow westward) and sunset (with the sun in the west, casting the shadow eastward). These shadows have opposite directions, so it may be that Crishna is describing signed infinities $\frac{\pm p}{0} = \infty_{\pm p}$. Now we get to the nub of the matter. If Crishna is describing signed infinities, then he will have multiple contradictions with Brahmegupta's sign conventions, most of which Bhascara accepts, when $\frac{0}{0} = 0$ is generated. But if Crishna is describing unsigned infinities and unsigned $\frac{0}{0}$, he will have no contradictions with Brahmegupta's trichotomous sign conventions, because unsigned infinity and unsigned $\frac{0}{0}$ fall into the tetrachotomous class of non-negative and non-zero and non-positive. We generously assume that Bhascara and his commentator, Crishna, describe a class of unsigned infinities $\frac{n_i}{0}$ and an unsigned $\frac{0}{0}$.

V16 *In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth.*

Verse V16 is problematic. We suppose that *inserted* means "added" and *extracted* means "subtracted," but What does "many" mean? "Many" cannot be an infinite fraction, as we now see.

With Addition A, the fracterm $\frac{n_1}{0}$ always transforms into $\frac{0}{0}$.

$$\frac{n_1}{0} + \frac{n_2}{0} = \frac{n_1 \times 0 + n_2 \times 0}{0 \times 0} = \frac{0}{0} \tag{22}$$

With Addition B, the fracterm $\frac{n_1}{0}$ sometimes transforms into $\frac{0}{0}$.

$$\frac{n_1}{0} - \frac{n_1}{0} = \frac{n_1 - n_1}{0} = \frac{0}{0}$$

However, if "many" means any positive or negative finite number or zero, then the verse is correct, as shown by Equation 21. This is further evidence that we have correctly identified the class of fractions that explains Bhascara's and Crishna's conception of the infinite.

Verse V16 also refers to God. We would like to know which Sanscrit word was used for "God." Are the algebraic properties of God consistent with the sign conventions? For example, can God absorb all positive and negative finite additions without change? Can God be made zero by multiplication with zero? Can God be made zero by the addition of other infinities, as in Equation 22, when we accept $\frac{0}{0} = 0$? Is the negative of God the same as God or the opposite of God? We wonder, did Bhascara intend his allusion to God to explain his arithmetic or to explain God?

We now define Bhascara's identification of fractions, with a non-zero numerator and zero denominator, as unsigned infinities with unboundedly large magnitudes. As usual, we take the earliest verse, here V14, as index.

V14.1 Definition: for all non-zero real n_i, the fracterm $\frac{n_i}{0}$ is unordered and its magnitude $|\frac{n_i}{0}|$ is unboundedly large.

V14.2 Definition: the fracterm $\frac{0}{0}$ is unordered.

2.3.3 Bhascara's Rehearsal of Brahmegupta

In the previous two subsections we recorded how Bhascara rehearsed Brahmegupta's arithmetic. Some of Brahmegupta's indexed ideas are just titles, which we do not map. These are: C31.1, C32.1, C34.1, C35.1. This leaves seven of Brahmegupta's substantive indexed ideas that are not mapped onto Bhascara's texts. These are: G3.1, G4.1, C31.5, C32.2, C32.3, C32.5, C32.6. We now comment on these.

G4.1 is a backward reference to G3.1. They have the same content, which is to convert a mixed fraction $i\frac{n}{d}$ into a vulgar fraction or fracterm $\frac{id+n}{d}$. Here all of i, n, d are positive integers. Such conversions are discussed in Bhascara's Lilavati, but it is easier and more instructive to derive this conversion as a theorem of his arithmetic.

Firstly, we note that the Sanscrit mathematicians write $i + \frac{n}{d}$ in a tableau, without writing the plus sign, +, or the horizontal bar, -, separating the numerator and denominator of a fracterm. Thus:

$$\begin{matrix} i & n \\ & d \end{matrix}$$

When we write the horizontal bar in the fraction, following the Western style, but omit the addition, following the Sanscrit style, we obtain the Western mixed fraction $i\frac{n}{d} = i + \frac{n}{d}$. We prove:

Theorem 2.2. $i + \frac{n}{d} = \frac{id+n}{d}$.

Proof.

$$\begin{aligned} i + \frac{n}{d} &= \frac{i}{1} + \frac{n}{d} \text{ by L36.3} \\ &= \frac{id + n1}{1d} \text{ by G2.2A} \\ &= \frac{id+n}{d} \end{aligned}$$

□

C31.5 is the additive identity $r - r = 0$. The history of mathematics teaches us that this identity was not lost between the seventh and twelfth centuries in either India or Arabia so it must be that Bhascara chose not to mention it, perhaps because it was a widely accepted part of arithmetic.

C32.2, C32.3, C32.5, and C32.6 are all aspects of signed arithmetic. Perhaps they are mentioned somewhere in the Lilavati or are consequences of the Vija-ganita, but we have not found them anywhere in Bhascara's texts.

2.3.4 Summary of Bhascara's Texts

Bhascara introduced an inchoate arithmetic of infinitesimal tuples, $\langle 0, r \rangle$, such that $r \times 0 = 0 \times r = \langle 0, r \rangle$ and $\frac{\langle 0, r \rangle}{0} = r$ for all real r. But this infinitesimal arithmetic is contradictory because it implies $\frac{0}{0} = 0$.

Bhascara implied that all fractions, $\frac{n_i}{0}$, with a non-zero real numerator, n_i, and zero denominator, have infinite magnitude, which we read as saying that Bhascara's infinities, $\frac{n_i}{0}$, are unsigned, because if they were signed, they would generate $\frac{0}{0} = 0$ which contradicts Brahmegupta's sign conventions, most of which Bhascara accepts.

Bhascara does not explicitly accept Brahmegupta's $\frac{0}{0} = 0$, though it is implicit, but not used, in his writings. We generously suppose that Bhascara did not accept $\frac{0}{0} = 0$ but, instead, left $\frac{0}{0}$ as an unsigned number.

3 Sanscrit Arithmetics

Brahmegupta, 628 CE, introduced a total paraconsistent lexical arithmetic of fractions that contains real arithmetic. His account of addition and subtraction is ambiguous. This splits his arithmetic into two arithmetics, one using Addition A and the other using Addition B. These give the same results for addition and subtraction of finite numbers, but generally give different results when both arguments have a zero denominator. Brahmegupta's arithmetics have three classes of fractions that contain a zero numerator and/or denominator: $\frac{n}{0}$, $\frac{0}{0}$, $\frac{0}{d}$. It is a theorem of Brahmegupta's arithmetics that $\frac{0}{d} = 0$, but it is not clear whether he was aware of this equality.

We credit Bhascara, 1150 CE, with identifying each, $\frac{n_i}{0}$, as an unsigned infinity, $\infty_{n_i} = \frac{n_i}{0}$, leaving $\frac{0}{0}$ as an unsigned fraction we call nullity, $\Phi = \frac{0}{0}$, despite the fact that it has only some of the algebraic properties of transreal nullity. In our model of Sanscrit arithmetic, nullity is an unordered fraction with the real number zero as numerator and denominator, but Sanscrit arithmetic departs from transreal arithmetic in rectification, cancellation, addition, and distributivity. When we take the infinities and nullity as unsigned, both of Brahmegupta's total paraconsistent arithmetics become total consistent arithmetics, and both contain real arithmetic which is, itself, consistent.

Bhascara also described an inchoate partial inconsistent lexical arithmetic of infinitesimal tuples $\langle 0, r \rangle$. Nonetheless, Bhascara's infinitesimal arithmetic contains real arithmetic.

Both Brahmegupta and Bhascara had limited access to computable real numbers. We use real-numbered numerators, denominators, and coefficients in the knowledge that our analysis applies to all of the numbers they had access to. Everywhere in this paper we let d_i, n_i, p_i, r_i, s_i be arbitrary real numbers such that: d_i and n_i are non-zero; p_i is positive, whence $-p_i$ is negative; finally r_i and s_i have arbitrary trichotomous signs. Fractions are generally written as $\frac{n_i}{d_i}$ and $\frac{r_i}{s_i}$. A general Sanscrit or Hindu number is written as $h_i = \frac{r_i}{s_i}$.

We say Brahmegupta's arithmetic is described by: operation G2.2A, G2.3A, G2.2B, G2.3B, G3.2, G4.2; sign convention C31 in Subsection 2.2.2, C32 in Subsection 2.2.3, C34 in Subsection 2.2.4, C35 in Subsection 2.2.5; Observation G2.4; Theorem 2.1; and Definition V14.1 and V14.2. This arithmetic is shared by Bhascara, except that Bhascara clearly used Addition B and did not accept $\frac{0}{0} = 0$. Bhascara also described an inchoate arithmetic of infinitesimal numbers at L44.8, L44.9, but he made little use of it. We do not consider it further.

We present theorems of the Sanscrit fractional arithmetics, excluding the infinitesimal arithmetic. The reader is free to take the parts that relate to Brahmegupta or Bhascara. Multiplication and division are common to all of their fractional arithmetics. We do not know if Brahmegupta used Addition A or Addition B, but Bhascara used Addition B.

Both Brahmegupta and Bhascara knew that $a - b = a + (-b)$, see C31.4 in Subsection 2.2.2 and Verse V3, whence it suffices to present addition without subtraction. They both also knew how to rewrite division in terms of multiplication. See Brahmegupta's G4.2 in Section 2.1.4 and Bhascara's L40 in Section 2.3.2. Whence we do not need to present division of fractions, though we do rehearse division of components and the reciprocal. We do not rehearse proofs involving only real numbers.

3.1 Multiplication

Multiplication is common to all of Brahmegupta's and Bhascara's arithmetics. Recall that we use the mnemonic in Definition 1.1, rehearsed above, for variable names everywhere in this paper, including in the following proofs.

Theorem 3.1. $\frac{0}{0} \times \frac{r}{s} = \frac{0}{0} = 0$. $(\Phi \times h = \Phi = 0)$

Proof.

$$\begin{aligned} \frac{0}{0} \times \frac{r}{s} &= \frac{0 \times r}{0 \times s} \text{ by G3.2} \\ &= \frac{0}{0} \text{ by C34.4, C34.5} \\ &= 0 \text{ by C35.3} \end{aligned}$$

□

Bhascara would stop at $\frac{0}{0} \times \frac{r}{s} = \frac{0}{0}$, but Brahmegupta would continue to $\frac{0}{0} \times \frac{r}{s} = 0$.

Theorem 3.2. $\frac{n}{0} \times \frac{0}{d} = \frac{0}{0} = 0$. $(\infty \times 0 = \Phi = 0)$

Proof.

$$\begin{aligned} \frac{n}{0} \times \frac{0}{d} &= \frac{n \times 0}{0 \times d} \text{ by G3.2} \\ &= \frac{0}{0} \text{ by C34.4} \\ &= 0 \text{ by C35.3} \end{aligned}$$

□

Bhascara would stop at $\frac{n}{0} \times \frac{0}{d} = \frac{0}{0}$, but Brahmegupta would continue to $\frac{n}{0} \times \frac{0}{d} = 0$.

Theorem 3.3. $\frac{n_1}{0} \times n_2 = \frac{n_1 \times n_2}{0}$. $(\infty_{n_1} \times n_2 = \infty_{n_1 \times n_2})$

Proof.

$$\begin{aligned} \frac{n_1}{0} \times n_2 &= \frac{n_1}{0} \times \frac{n_2}{1} \text{ by G2.4} \\ &= \frac{n_1 \times n_2}{0 \times 1} \text{ by G3.2} \\ &= \frac{n_1 \times n_2}{0} \text{ by C34.2, C34.3, C34.4} \end{aligned}$$

□

Theorem 3.4. $\frac{n_1}{0} \times \frac{n_2}{0} = \frac{n_1 \times n_2}{0}$. $(\infty_{n_1} \times \infty_{n_2} = \infty_{n_1 \times n_2})$

Proof.

$$\begin{aligned} \frac{n_1}{0} \times \frac{n_2}{0} &= \frac{n_1 \times n_2}{0 \times 0} \text{ by G3.2} \\ &= \frac{n_1 \times n_2}{0} \text{ by C34.2, C34.3, C34.5} \end{aligned}$$

□

Remark 3.5. *Multiplication of fracterms $\frac{r_i}{s_i}$ is associative and commutative. This follows from the corresponding properties of real r_i, s_i. Similarly $\frac{0}{0}$ is absorptive over multiplication and therefore also over division.*

3.2 Division

Division is common to all of Brahmegupta's and Bhascara's arithmetics. Sanscrit arithmetic begins from integral numbers written in decimal notation. These are a subset of the real numbers. Fractions are composed from them by division.

Remark 3.6. *Definition 1.6 says: components r and s are divided by the operation "÷" as:*

$$r \div s = \frac{r}{s}$$

Remark 3.7. *Definition 1.7 says: a fracterm $\frac{r}{s}$ is reciprocated by the operation "-1" as:*

$$\left(\frac{r}{s}\right)^{-1} = \frac{s}{r}$$

The choice of zero's denominator, d_i in $\frac{0}{d_i}$, makes a difference to the reciprocal, $\frac{d_i}{0}$, because in the Addition B subtraction $\frac{d_i}{0} - \frac{d_j}{0} = \frac{d_i - d_j}{0}$, when $d_i \neq d_j$ the result is an infinity, $\frac{d_k}{0}$, but when $d_i = d_j$ the result is $\frac{0}{0}$, which is not an infinity. This indeterminacy is avoided if we always specify the denominator of zero whenever we operate on zero, which we must do when calculating with fractions. Thus, we obtain unique solutions even in the presence of inconsistency among the axioms.

3.3 Addition A

For all additions, following Addition A, and for all additions without a common denominator, following Addition B, it is the case that G2.2A and the following theorems hold.

Theorem 3.8. $\frac{0}{0} + \frac{r}{s} = \frac{0}{0} = 0$. $(\Phi + h = \Phi = 0)$.

Proof.

$$\begin{aligned}\frac{0}{0} + \frac{r}{s} &= \frac{0 \times s + r \times 0}{0 \times s} \text{ by G2.2A}\\ &= \frac{0}{0} \text{ by C34.4, C34.5, C31.8}\\ &= 0 \text{ by C35.3}\end{aligned}$$

□

Bhascara would stop at $\frac{0}{0} + \frac{r}{s} = \frac{0}{0}$, but Brahmegupta would continue to $\frac{0}{0} + \frac{r}{s} = 0$.

Theorem 3.9. $\frac{n}{0} + \frac{0}{d} = \frac{n \times d}{0}$. $(\infty_n + 0_d = \infty_{n \times d})$.

Proof.

$$\begin{aligned}\frac{n}{0} + \frac{0}{d} &= \frac{n \times d + 0 \times 0}{0 \times d} \text{ by G2.2A}\\ &= \frac{n \times d}{0} \text{ by C34.2, C34.3, C34.4, C34.5, C31.6, C31.7}\end{aligned}$$

□

Theorem 3.10. $\frac{n_1}{0} + \frac{n_2}{d_2} = \frac{n_1 \times d_2}{0}$ $(\infty_{n_1} + \frac{n_2}{d_2} = \infty_{n_1 \times d_2})$

Proof.

$$\begin{aligned}\frac{n_1}{0} + \frac{n_2}{d_2} &= \frac{n_1 \times d_2 + n_2 \times 0}{0 \times d_2} \text{ by G2.2A}\\ &= \frac{n_1 \times d_2}{0} \text{ by C34.4, C31.6, C31.7}\end{aligned}$$

□

Theorem 3.11. $\frac{n_1}{0} + \frac{n_2}{0} = \frac{0}{0} = 0$. $(\infty + \infty = \Phi = 0)$.

Proof.

$$\begin{aligned}\frac{n_1}{0} + \frac{n_2}{0} &= \frac{n_1 \times 0 + n_2 \times 0}{0 \times 0} \text{ by G2.2A} \\ &= \frac{0}{0} \text{ by C34.4, C34.5, C31.8} \\ &= 0 \text{ by C35.3}\end{aligned}$$

□

Bhascara would stop at $\frac{n_1}{0} + \frac{n_2}{0} = \frac{0}{0}$, but Brahmegupta would continue to $\frac{n_1}{0} + \frac{n_2}{0} = 0$.

Remark 3.12. *In Addition A, the addition of fracterms $\frac{r_i}{s_i}$ is associative and commutative. This follows from the corresponding properties of the real r_i, s_i. Similarly $\frac{0}{0}$ is absorptive over all Addition A sums and differences.*

3.4 Addition B

For all additions, following Addition B, with a common denominator, both G2.2B and the following theorem hold.

Theorem 3.13. $\frac{r_1}{0} + \frac{r_2}{0} = \frac{r_1+r_2}{0} \in \{\infty_{r_1+r_2 \neq 0}, \Phi, 0\}$.

Proof.

$$\begin{aligned}\frac{r_1}{0} + \frac{r_2}{0} &= \frac{r_1 + r_2}{0} \text{ by G2.2B} \\ &= \begin{cases} \infty_{r_1+r_2} & : r_1 + r_2 \neq 0 \text{ by V14.1} \\ \Phi & : r_1 + r_2 = 0 \text{ by C31.5} \\ 0 & : r_1 + r_2 = 0 \text{ by C35.3} \end{cases}\end{aligned}$$

□

When $r_1 + r_2 = 0$, Bhascara would stop at $\frac{r_1}{0} + \frac{r_2}{0} = \frac{0}{0}$, but Brahmegupta would continue to $\frac{r_1}{0} + \frac{r_2}{0} = 0$.

Remark 3.14. *In Addition B, the addition of fracterms, $\frac{r_i}{s}$, with a common denominator, s, is associative and commutative. This follows from the corresponding properties of the real r_i, s_i. Similarly $\frac{0}{0}$ is non-absorptive in Addition B.*

Theorem 3.15. *In Addition B, when fracterms, $\frac{r_i}{s_i}$, with some common and some distinct denominators, s_i, are summed, the result is generally non-associative and it is, therefore, generally non-commutative.*

In the following proof, we use the arithmetic established above without citing foundational material.

Proof. Addition is associative if and only if $(a+b)+c = a+(b+c)$. We now give one of two counter examples, with three terms, not counting the third commutative permutation of this example.

$$\begin{aligned}
\text{Let LHS } &= \left(\frac{r_1}{0} + \frac{r_2}{0}\right) + \frac{r_3}{d_3} \\
&= \frac{r_1 + r_2}{0} + \frac{r_3}{d_3} \\
&= \frac{(r_1 + r_2) \times d_3 + r_3 \times 0}{0 \times d_3} \\
&= \frac{(r_1 + r_2) \times d_3}{0}
\end{aligned}$$

$$\begin{aligned}
\text{Let RHS} &= \frac{r_1}{0} + \left(\frac{r_2}{0} + \frac{r_3}{d_3}\right) \\
&= \frac{r_1}{0} + \frac{r_2 \times d_3 + r_3 \times 0}{0 \times d_3} \\
&= \frac{r_1}{0} + \frac{r_2 \times d_3}{0} \\
&= \frac{r_1 + (r_2 \times d_3)}{0}
\end{aligned}$$

This sum is non-associative exactly when LHS $\neq$ RHS. This occurs, for example, when $r_1 = r_2 = 1$ and $d_3 = -1$, because LHS $= \frac{(1+1)\times(-1)}{0} = \frac{-2}{0} = \infty_{-2} \neq$ RHS $= \frac{1+1\times(-1)}{0} = \frac{0}{0} = 0$. Here, Bhascara would stop at RHS $= \frac{0}{0}$, but Brahmegupta would continue to RHS $= 0$. □

3.5 Distributivity

From here onward, we use the modern mathematical convention of implicit multiplication, so that $ab = a(b) = a \times b$.

Theorem 3.16. *Multiplication distributes over Addition B of fracterms with a common denominator.*

Proof. Because Brahmegupta's lexical arithmetic contains real arithmetic, it is sufficient to prove the theorem when $s_2 = s_3 = 0$.

Recall that multiplication distributes over addition exactly when $\frac{r_1}{s_1}(\frac{r_2}{s_2} + \frac{r_3}{s_3}) = \frac{r_1 r_2}{s_1 s_2} + \frac{r_1 r_3}{s_1 s_3}$. Now:

$$\begin{aligned}
\text{Let LHS} &= \frac{r_1}{s_1}\left(\frac{r_2}{0} + \frac{r_3}{0}\right) \\
&= \frac{r_1}{s_1}\left(\frac{r_2 + r_3}{0}\right) \\
&= \frac{r_1(r_2 + r_3)}{0}
\end{aligned}$$

$$
\begin{aligned}
\text{Let RHS} &= \frac{r_1 r_2}{s_1 0} + \frac{r_1 r_3}{s_1 0} \\
&= \frac{r_1 r_2 + r_1 r_3}{s_1 0} \\
&= \frac{r_1(r_2 + r_3)}{0}
\end{aligned}
$$

Therefore LHS = RHS. □

Theorem 3.17. *Multiplication distributes over Addition A when $s_1 \neq 0$.*

Proof.

$$
\begin{aligned}
\text{Let LHS} &= \frac{r_1}{s_1}\left(\frac{r_2}{s_2} + \frac{r_3}{s_3}\right) \\
&= \frac{r_1}{s_1}\left(\frac{r_2 s_3 + s_2 r_3}{s_2 s_3}\right) \\
&= \frac{r_1 r_2 s_3 + r_1 s_2 r_3}{s_1 s_2 s_3}
\end{aligned}
$$

$$
\begin{aligned}
\text{Let RHS} &= \frac{r_1 r_2}{s_1 s_2} + \frac{r_1 r_3}{s_1 s_3} \\
&= \frac{r_1 s_1 r_2 s_3 + r_1 s_1 s_2 r_3}{s_1 s_1 s_2 s_3} \\
&= \frac{s_1}{s_1} \times \frac{r_1 r_2 s_3 + r_1 s_2 r_3}{s_1 s_2 s_3} \\
&= 1 \times \frac{r_1 r_2 s_3 + r_1 s_2 r_3}{s_1 s_2 s_3} \text{ when } s_1 \neq 0 \\
&= \frac{r_1 r_2 s_3 + r_1 s_2 r_3}{s_1 s_2 s_3}
\end{aligned}
$$

Therefore LHS = RHS when $s_1 \neq 0$. □

Theorem 3.18. *Multiplication does not distribute over Addition A when $s_1 = 0$ & $r_1(r_2 s_3 + s_2 r_3) \neq 0$.*

Proof. Taking up from the proof of Theorem 3.17

$$
\begin{aligned}
\text{LHS} &= \frac{r_1 r_2 s_3 + r_1 s_2 r_3}{0 s_2 s_3} \text{ when } s_1 = 0 \\
&= \frac{r_1(r_2 s_3 + s_2 r_3)}{0} \\
&= \infty_{r_1(r_2 s_3 + s_2 r_3) \neq 0}
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS} &= \frac{r_1 0 r_2 s_3 + r_1 0 s_2 r_3}{0(0 s_2 s_3)} \text{ when } s_1 = 0 \\
&= \frac{0}{0} \text{ following Bhascara} \\
&= 0 \text{ following Brahmegupta}
\end{aligned}
$$

Therefore, LHS $\neq$ RHS when $s_1 = 0$ & $r_1(r_2s_3 + s_2r_3) \neq 0$. □

Theorem 3.17 says that multiplication distributes over Addition A when $s_1 \neq 0$. Theorem 3.18 says that multiplication does not distribute over Addition A when $s_1 = 0$ & $r_1(r_2s_3 + s_2r_3) \neq 0$. These two theorems are consistent, because they are disjoint in s_1. In summary, Theorem 3.18 categorises the whole of non-distributivity over Addition A.

Theorem 3.16 says that multiplication distributes over Addition B, so the only non-distribution is the non-distribution in Addition A, which is categorised in Theorem 3.18. Thus, when using Addition A, with any denominators, and when using Addition B, with distinct denominators, multiplication distributes over all additions, unless $s_1 = 0$ & $r_1(r_2s_3 + s_2r_3) \neq 0$.

3.6 Bottom

The modern reader might wonder if it is possible to use bottom, $\bot$, to characterise division by zero in the Sanscrit fractional arithmetics?

Recall that bottom is totally absorptive, such that all additions, subtractions, multiplications, and divisions, with bottom as an argument, produce bottom as a result.

The fracterm $\frac{0}{0}$ has the required absorptivity in multiplication, see Theorem 3.1. Whence $\frac{0}{0}$ is absorptive in division. The fracterm $\frac{0}{0}$ is also absorptive in Addition A, see Theorem 3.8, so we could make the identification $\frac{0}{0} = \bot$ in Addition A, but not in the more popular Addition B, because, for example, $\frac{1}{0} + \frac{0}{0} = \frac{1+0}{0} = \frac{1}{0} \neq \frac{0}{0} = \bot$.

Nor can we take $\frac{n_i}{0} = \bot$, in any of the Sanscrit fractional arithmetics, because, for example, $1 \div \bot = \frac{1}{1} \div \frac{n_i}{0} = \frac{1}{1} \times \frac{0}{n_1} = \frac{1 \times 0}{1 \times n_i} = \frac{0}{n_1} = 0 \neq \bot$.

In short, the only opportunity to use bottom to explain division by zero in the Sanscrit fractional arithmetics is in those arithmetics using Addition A, but not Addition B.

3.7 Paraconsistency

We say that an inconsistent arithmetic is "paraconsistent" if each calculation is deterministic in the sense that it gives a single result. We prove that such arithmetics exist by giving a silly example. Given a sound axiomatic specification of a deterministic arithmetic, we conjoin two axioms to it. Axiom +1: All numbers are colourless. Axiom +2: 7 is blue. Now the axioms of the arithmetic are inconsistent, but all calculations remain deterministic.

The Sanscrit arithmetics just presented all have inconsistent handling of the fraction $\frac{0}{0}$. Brahmegupta asserts $\frac{0}{0} = 0$, which contradicts his sign conventions. Bhascara does not explicitly accept $\frac{0}{0} = 0$, but he can independently calculate $\frac{0}{0} = \frac{0 \times 0}{0} = \frac{\langle 0,0 \rangle}{0} = 0$, which contradicts Brahmegupta's sign conventions that Bhascara accepts. In particular, $\frac{0}{0}$ is absorptive in Addition A, with any denominators, and in Addition B, with distinct denominators; but 0 is asserted, by both Brahmegupta and Bhascara, to be non-absorptive over addition. Thus, the absorptivity of $\frac{0}{0}$ is inconsistent with the additive non-absorptivity of 0, which breaks the sign conventions. Nonetheless, if we always specify the denominator of a zero numerator and, more generally, if we always specify the numerator and denominator of a fraction, then every component of a fraction is specified uniquely, and, as the arithmetic of components is deterministic, the arithmetic of fractions is deterministic. Thus, by always giving the numerator and denominator, we make all of the Sanscrit arithmetics paraconsistent. Put another way, the Sanscrit fractional arithmetics are paraconsistent because they avoid interchanging 0 and $\frac{0}{0}$.

This paraconsistency is meaningful. Brahmegupta inherited the algorithms of pre-existing integral and fractional partial arithmetics, which he generalised into algorithms for total fractional arithmetics. He then asserted many properties of his arithmetics, but some of these assertions are overspecified and contradict his algorithms. Thus, when the numerators and denominators are specified, his algorithms are deterministic, but his assertions are contradictory. Whence we say that Brahmegupta's, and hence Bhascara's, arithmetics are paraconsistent.

However, we credit Bhascara with making the infinities unsigned and, as he did not mention $\frac{0}{0}$, we take it that this fraction is also unsigned. This generous interpretation makes Bhascara's fractional arithmetic consistent, because it does not break the sign conventions. This arrangement also makes both of Brahmegupta's fractional arithmetics consistent, whence we generously credit Bhascara with making all of the Sanscrit fractional arithmetics consistent.

In the early development of any scientific system, determinism makes the system useful and provides a basis for logical development. In these early stages, we expect both inconsistency and partiality. Before an exact specification is obtained, we expect under specification, leading to partiality, as in Brahmegupta's and Bhascara's handling of the unsigned fracterms $\frac{n_i}{0}$, and we expect overspecification, leading to contradiction, as in Brahmegupta's $\frac{0}{0} = 0$.

A worse situation arises in the modern specification of floating-point arithmetic, which is a modern totalisation of a limited precision, discrete approximation to real arithmetic. The IEEE 754 Standard for Floating-Point Arithmetic [1], [2], [3] is a deterministic manipulation of floating-point bits, but it is non-logical, because $NaN_i \neq NaN_i$ breaks the Law of Identity and, even if the arithmetic were logical, it would be inconsistent, because $-0 = 0$ and $\frac{1}{-0} \neq \frac{1}{0}$, and even if the arithmetic were consistent it would be incoherent, because each abstract NaN_i has two bit patterns, such that each abstract NaN_i orders before and after all floating-point bit patterns, including itself, and, contradicting what has just been said, each abstract NaN_i and its two bit patterns are unordered [9]. It is extraordinary that in the twentieth and twenty-first centuries CE, computer scientists produced a less sound total arithmetic than Brahmegupta did in the seventh century CE. It is even more extraordinary that today's mathematicians use floating-point arithmetic without raising objections to its unsoundness.

We say that a system that is deterministic but non-logical is "paralogical." Thus, the IEEE standard describes a paralogical total arithmetic. This is less sound than Brahmegupta's and Bhascara's paraconsistent and consistent total arithmetics.

We recommend that all computer arithmetics are founded on transreal arithmetic, which is a logical consistent total arithmetic [4] that supports calculus [5], [20], [21], [22], [23] and mathematical physics [6].

When division by zero is outlawed, all of the Sanscrit arithmetics we have discussed give a lexical description of real arithmetic, which accesses as much of real arithmetic as Sanscrit mathematicians could perform. This was the arithmetic of some computable real numbers.

4 Future Work

We have no immediate plans to do more work on Sanscrit mathematics, but we would be happy to discuss research with others.

We would like to know what mathematics was known to Sanscrit mathematicians before Brahmegupta? In particular, what was known by armchair reasoning and what was empirically tested by using algorithms, such as the Pulverizer?

Have we correctly understood the layout of mixed fractions in Sanscrit mathematics? What chain of historical evidence bears on its transmission to the West?

Do the Sanscrit texts support our interpretation that zero was not considered to be a finite number?

Do the Sanscrit texts support our interpretation that Brahmegupta described the fractions $\frac{n_i}{0}$, $\frac{0}{0}$, $\frac{0}{d_i}$?

Do the Sanscrit texts support our interpretation that Brahmegupta described two forms of arithmetic, one with Addition A and the other with Addition B? Is there evidence as to what form of addition Brahmegupta used?

Do the Sanscrit texts make clear whether Bhascara's infinities were signed or unsigned?

Do the Sanscrit texts make clear whether Bhascara thought that $\frac{0}{0}$ is zero or infinite or did he simply ignore $\frac{0}{0}$?

Do the Sanscrit texts support our interpretaion that Bhascara used the tuple $\langle 0, h \rangle$ to describe infinitesimal numbers?

If some or all of these questions are answered, we might offer advice on routes to totalising Sanscrit arithmetic that we have not presented here.

The identification of totalisations of Sanscrit arithmetics might place Sanscrit arithmetics in a conceptual tree of the development of total arithmetics, independently of the historical role of Sanscrit arithmetics in the development of partial real arithmetic and totalisations of it, such as, but not limited to, transreal arithmetic.

We wonder whether the Western world should acknowledge the seminal role of Sanscrit mathematicians by renaming the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 as "Sanscrit numerals," "Hindu numerals," or "Indian numerals?" Are there historical, political, or religious reasons to prefer one name over another? Perhaps we should call them "Sanscrit-Arabic numerals" or "Hindu-Arabic numerals" or "Indian-Arabic numerals" to reflect the path by which they were transmitted to us in the West?

5 Conclusion

We have reviewed the book "ALGEBRA WITH ARITHMETIC AND MENSURATION, FROM THE SANSCRIT OF BRAHMEGUPTA AND BHASCARA" by Henry Thomas Colebrooke, 1817 CE [16].

The book presents a history of how mathematics diffused from India, via Arabia, to Europe. Colebrooke laments the opportunities to advance Western mathematics that were lost to the late arrival of Sanscrit mathematics in Europe. This is ironic because Colebrooke did not notice that the Sanscrit arithmetics he described are total arithmetics, a possibility that was first revealed to the Western world one hundred and forty years after Colebrooke's publication. It is not enough that mathematics are made known, they must also be understood and appreciated!

We discover, in Colebrooke's book, that there were two sources of algebraic structure that were available to inform Brahmegupta and the Sanscrit mathematicians. Firstly, there was armchair reasoning about distances and other physical or social metrics, such as wealth. Secondly, there was empirical verification of the algebraic structure of signed and zero quantities built into algorithms, such as the Pulverizer.

We find that Brahmegupta, 628 CE, described two total paraconsistent lexical arithmetics that contain real arithmetic. We generously credit Bhascara, 1150 CE, with making these arithmetics consistent. However, Bhascara also introduced an inchoate partial inconsistent arithmetic of infinitesimal tuples.

We observe that Brahmegupta's and Bhascara's total fractional arithmetics are more logically sound than the IEEE 754 Standard for Floating-Point Arithmetic. We recommend that transreal arithmetic, being a total consistent arithmetic that extends calculus and mathematical physics, is used as a foundation for computer arithmetics.

Informed by our experience of the bottom-up analysis of Colebrooke's book, we propose a collection of lexical operations that can be used in the top-down analysis of both partial and total arithmetics that are descendants of Sanscrit arithmetic.

If our translation from 19th Century to 21st Century English reveals anything of interest, we hope Sanscrit scholars will consider retranslating parts of the original Sanscrit texts.

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