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Construction of the Transreal Numbers from Hyperreal Numbers

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Abstract

We construct the transreal numbers and arithmetic from subsets of hyperreal numbers. In possession of this construction, we propose a contextual interpretation of the transreal arithmetical operations as vector transformations.

1 Introduction

The first proof of the consistency of the axioms of transreal arithmetic is a machine proof [1]. Further to this, there are a number of human constructions of the transreal numbers from widely accepted sets of numbers. This establishes the consistency of the transreals relative to the accepted sets. In [2] transreal numbers arise as trans-Dedekind cuts on rational numbers, thereby establishing the consistency of the transreals relative to the Dedekind cuts and thence to the rationals. In [4] transreal numbers arise as equivalence classes of ordered pairs of real numbers, thereby establishing the consistency of the transreals relative to the reals. There are two other similar constructions: in [5] transreal numbers arise as a subset of the transcomplex numbers which, in turn, arise as equivalence classes of ordered pairs of complex numbers, thereby establishing the consistency of the transreals relative to the complexes; and in a paper in preparation [6] transreal numbers arise as a subset of the transquaternion numbers which, in turn, arise as equivalence classes of ordered pairs of quaternion numbers, thereby establishing the consistency of the transreals relative to the quaternions. In the present paper we propose a further way to construct the transreal numbers. We construct the transreal numbers as subsets of hyperreal numbers. In this way, the consistency of transreal numbers is also justified by their consistency with the hyperreal numbers.

In [3] we wondered if, like the real numbers, the transreal numbers can be interpreted as vectors. In that paper we talked about the transreal numbers positive infinity and negative infinity being vectors whose magnitudes are greater than any real number. However, the set of real numbers cannot support this interpretation since, of course, there is no real number greater than all real numbers. Another important feature of that interpretation was that the vectors which play the roles of the infinite transreal numbers do not have specific infinite magnitudes but do have a range of possible infinite magnitudes. We now find the set of hyperreal numbers is very convenient for these purposes. There are hyperreal numbers that are greater than all real numbers, indeed there are many hyperreal numbers with this property. Hence we can use the infinitely many infinite hyperreal numbers to be the magnitudes of vectors which play the role of the transreal infinities. Furthermore the set of hyperreal numbers contains the multiplicative inverse of all infinite hyperreal numbers, which allows us to define total division in our new interpretation of the transreals. We explain the details on this interpretation in the section after the construction.

We define nullity as the whole set of the hyperreal numbers, positive infinity as the set of positive infinite hyperreal numbers, negative infinity as the set of negative infinite hyperreal numbers, and each real number as its monad. Note that every transreal number is a set of hyperreal numbers. Then we define the arithmetical operations and order relation, in natural way, on these sets.

2 Preliminaries on Hyperreal Numbers

As is known [7]:

- The set of *hyperreal numbers*, ${}^*\mathbb{R}$, is an ordered field which properly contains the set of real numbers as a subfield. In other words: ${}^*\mathbb{R}$ is an ordered field, $\mathbb{R} \subsetneq {}^*\mathbb{R}$, and addition and multiplication of ${}^*\mathbb{R}$ extend addition and multiplication of $\mathbb{R}$.
- There are numbers $\delta \in {}^*\mathbb{R}$ such that $|\delta| < r$ for all positive $r \in \mathbb{R}$. These numbers are called *infinitesimal numbers*. Zero, 0, is the only infinitesimal number which is a real number. The subset of infinitesimal numbers is denoted by ${}^*\mathbb{R}_0$.
- There are numbers $\omega \in {}^*\mathbb{R}$ such that $r < |\omega|$ for all positive $r \in \mathbb{R}$. These numbers are called *infinite numbers*. The subset of negative infinite numbers is denoted by ${}^*\mathbb{R}^-_\infty$ and the subset of positive infinite numbers is denoted by ${}^*\mathbb{R}^+_\infty$.

As consequence of the above:

1. If $\delta, \varepsilon \in {}^*\mathbb{R}_0$ then $\delta + \varepsilon \in {}^*\mathbb{R}_0$.
2. If $\delta, \varepsilon \in {}^*\mathbb{R}_0$ then $\delta\varepsilon \in {}^*\mathbb{R}_0$.
3. If $x \in \mathbb{R}$ and $\delta \in {}^*\mathbb{R}_0$ then $x\delta \in {}^*\mathbb{R}_0$.
4. For all $x, y \in \mathbb{R}$, $x < y$ if and only if $x+\delta < y+\varepsilon$ for every $\delta, \varepsilon \in {}^*\mathbb{R}_0$.
5. $\omega \in {}^*\mathbb{R}^+_\infty$ if and only if $-\omega \in {}^*\mathbb{R}^-_\infty$.

6. $\delta \in {}^*\mathbb{R}_0$ and $\delta > 0$ if and only if $\frac{1}{\delta} \in {}^*\mathbb{R}^+_\infty$.
7. $\delta \in {}^*\mathbb{R}_0$ and $\delta < 0$ if and only if $\frac{1}{\delta} \in {}^*\mathbb{R}^-_\infty$.
8. If $x \in \mathbb{R}$ and $\omega \in {}^*\mathbb{R}^+_\infty$ then $x + \omega \in {}^*\mathbb{R}^+_\infty$.
9. If $\delta \in {}^*\mathbb{R}_0$ and $\omega \in {}^*\mathbb{R}^+_\infty$ then $\delta + \omega \in {}^*\mathbb{R}^+_\infty$.
10. If $\upsilon \in {}^*\mathbb{R}^+_\infty$ and $\omega \in {}^*\mathbb{R}^+_\infty$ then $\upsilon + \omega \in {}^*\mathbb{R}^+_\infty$.
11. If $x \in \mathbb{R}$ and $\omega \in {}^*\mathbb{R}^-_\infty$ then $x + \omega \in {}^*\mathbb{R}^-_\infty$.
12. If $\delta \in {}^*\mathbb{R}_0$ and $\omega \in {}^*\mathbb{R}^-_\infty$ then $\delta + \omega \in {}^*\mathbb{R}^-_\infty$.
13. If $\upsilon \in {}^*\mathbb{R}^-_\infty$ and $\omega \in {}^*\mathbb{R}^-_\infty$ then $\upsilon + \omega \in {}^*\mathbb{R}^-_\infty$.
14. If $x \in \mathbb{R}$ and $x < 0$ and $\omega \in {}^*\mathbb{R}^+_\infty$ then $x\omega \in {}^*\mathbb{R}^-_\infty$.
15. If $x \in \mathbb{R}$ and $x > 0$ and $\omega \in {}^*\mathbb{R}^+_\infty$ then $x\omega \in {}^*\mathbb{R}^+_\infty$.
16. If $x \in \mathbb{R}$ and $x < 0$ and $\omega \in {}^*\mathbb{R}^-_\infty$ then $x\omega \in {}^*\mathbb{R}^+_\infty$.
17. If $x \in \mathbb{R}$ and $x > 0$ and $\omega \in {}^*\mathbb{R}^-_\infty$ then $x\omega \in {}^*\mathbb{R}^-_\infty$.
18. If $\upsilon \in {}^*\mathbb{R}^-_\infty$ and $\omega \in {}^*\mathbb{R}^+_\infty$ then $\upsilon\omega \in {}^*\mathbb{R}^-_\infty$.
19. If $\upsilon \in {}^*\mathbb{R}^+_\infty$ and $\omega \in {}^*\mathbb{R}^+_\infty$ then $\upsilon\omega \in {}^*\mathbb{R}^+_\infty$.

3 Transreal from Hyperreal Numbers

Definition 1. Let $T := \big\{\{\alpha + \delta;\ \delta \in {}^*\mathbb{R}_0\};\ \alpha \in \mathbb{R}\big\} \bigcup \big\{{}^*\mathbb{R}, {}^*\mathbb{R}^-_\infty, {}^*\mathbb{R}^+_\infty\big\}$.

Notice that $A \subset {}^*\mathbb{R}$ for every $A \in T$.

Definition 2. Given $A, B \in T$ let us define:

a) (addition) $A \oplus B := \{a + b;\ a \in A \text{ and } b \in B\}$.
b) (opposite) $\ominus A := \{-a;\ a \in A\}$.
c) (subtraction) $A \ominus B := A \oplus (\ominus B)$.
d) (multiplication) $A \otimes B := \{ab;\ a \in A \text{ and } b \in B\}$.
e) (reciprocal) $({}^*\mathbb{R}_0)^{\text{㊀1}} := \{|a|^{-1};\ a \in {}^*\mathbb{R}_0 \setminus \{0\}\}$,
$({}^*\mathbb{R})^{\text{㊀1}} := \{a^{-1};\ a \in {}^*\mathbb{R} \setminus \{0\}\} \cup \{0\}$,
$\big({}^*\mathbb{R}^-_\infty\big)^{\text{㊀1}} := \{a^{-1};\ a \in {}^*\mathbb{R}^-_\infty\} \cup \{-a^{-1};\ a \in {}^*\mathbb{R}^-_\infty\} \cup \{0\}$,
$\big({}^*\mathbb{R}^+_\infty\big)^{\text{㊀1}} := \{a^{-1};\ a \in {}^*\mathbb{R}^+_\infty\} \cup \{-a^{-1};\ a \in {}^*\mathbb{R}^+_\infty\} \cup \{0\}$ and
$A^{\text{㊀1}} := \{a^{-1};\ a \in A\}$, if $A \notin \{{}^*\mathbb{R}_0, {}^*\mathbb{R}, {}^*\mathbb{R}^-_\infty, {}^*\mathbb{R}^+_\infty\}$.
f) (division) $A \oslash B := A \otimes B^{\text{㊀1}}$.

Notice that the arithmetic is defined in a natural way. Each one of the operations between sets is defined by the respective operation between their elements. The reciprocal definition needs side conditions but it is still defined on sets by the reciprocal of their elements. Notice that for all $A \in T$ the reciprocal of A, in the new sense proposed here, is defined in terms of the reciprocal, in the usual sense, of the elements of A. However, in order to talk about reciprocal in the usual sense we do need the usual exception of the reciprocal of zero. Furthermore zero is not the usual reciprocal of any hyperreal number, because of this the reciprocals of ${}^*\mathbb{R}$, ${}^*\mathbb{R}^-_\infty$ and ${}^*\mathbb{R}^+_\infty$ would have a puncture. So we join zero to these in order to plug the puncture.

Definition 3. Let arbitrary $A, B \in T$. We say that $A \prec B$ if and only if $a < b$ for all $a \in A$ and all $b \in B$. Furthermore we say that $A \preceq B$ if and only if $A \prec B$ or $A = B$.

Notice that the relation $\preceq$ is an order relation on T.

The following theorem assures us that, in an appropriate sense, $\mathbb{R}$ is subset of T.

Theorem 4. The set $R := \big\{\{a + \delta;\ \delta \in {}^*\mathbb{R}_0\};\ a \in \mathbb{R}\big\}$ is a complete ordered field.

Proof. The result follows from the fact that $\pi : \mathbb{R} \to R$, $\pi(t) = \{t + \delta;\ \delta \in {}^*\mathbb{R}_0\}$ is bijective and, for any $t, s \in \mathbb{R}$,

i) $\pi(t) \oplus \pi(s) = \pi(t + s)$,

ii) $\pi(t) \otimes \pi(s) = \pi(ts)$ and

iii) $\pi(t) \preceq \pi(s)$ if and only if $t \leq s$,

and from the fact that $\mathbb{R}$ is a complete ordered field. □

Remark 5. Since π is an isomorphism of complete ordered fields between R and $\mathbb{R}$, we can say that R is a "copy" of $\mathbb{R}$ in T. Therefore let us abuse language and notation: henceforth R will be denoted by $\mathbb{R}$ and will be called the set of real numbers and each $\{\alpha + \delta;\ \delta \in {}^*\mathbb{R}_0\} \in R$ will be denoted, simply, by α and will be called a real number. In this sense we can say that $\mathbb{R} \subset T$ and we can replace the symbols $\oplus$, $\otimes$, $\prec$ and $\preceq$, respectively, by $+$, $\times$, $<$ and $\leq$. Furthermore, if $\alpha \in \mathbb{R}$ so $\ominus\alpha = \ominus\{\alpha + \delta;\ \delta \in {}^*\mathbb{R}_0\} = \{-\alpha - \delta;\ \delta \in {}^*\mathbb{R}_0\} = \{-\alpha + \delta;\ \delta \in {}^*\mathbb{R}_0\} = -\alpha$ and if $\alpha \neq 0$ so $\alpha^{\text{(-1)}} = \{\alpha + \delta;\ \delta \in {}^*\mathbb{R}_0\}^{\text{(-1)}} = \{(\alpha + \delta)^{-1};\ \delta \in {}^*\mathbb{R}_0\} = \{\alpha^{-1} + \delta;\ \delta \in {}^*\mathbb{R}_0\} = \alpha^{-1}$. Thus we replace the symbols $\ominus$, (-1) and $\oplus\!\!\!\!\div$ respectively, by $-$, ${}^{-1}$ and $\div$.

Definition 6. Let us define and denote *nullity*, *negative infinity* and *infinity*, respectively, by $\Phi := {}^*\mathbb{R}$, $-\infty := {}^*\mathbb{R}^-_\infty$ and $\infty := {}^*\mathbb{R}^+_\infty$. Let us refer to the elements of T as *transreal numbers*, thus T will be the *set of transreal numbers*. Let us denote $\mathbb{R}^T := T$ whence

$$\mathbb{R}^T = \mathbb{R} \cup \{\Phi, -\infty, \infty\}.$$

The next theorem sets out transreal arithmetic and ordering.

Theorem 7. For each $x \in \mathbb{R}^T$, it follows that:

a) $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

b) $x < \Phi$ does not hold and $\Phi < x$ does not hold.

c) $-\Phi = \Phi$.

d) $-(-\infty) = \infty$.

e) $-(\infty) = -\infty$.

f) $0^{-1} = \infty$.

g) $\Phi^{-1} = \Phi$.

h) $(-\infty)^{-1} = 0$.

i) $\infty^{-1} = 0$.

j) $\Phi + x = \Phi$.

k) $\infty + (-\infty) = \Phi$.

l) $\infty + \Phi = \Phi$.

m) $\infty + \infty = \infty$.

n) $\infty + x = \infty$ for all $x \in \mathbb{R}$.

o) $-\infty + \infty = \Phi$.

p) $-\infty + \Phi = \Phi$.

q) $-\infty + (-\infty) = -\infty$.

r) $-\infty + x = -\infty$ for all $x \in \mathbb{R}$.

s) $\Phi \times x = \Phi$.

t) $\infty \times 0 = \Phi$.

u) $\infty \times \Phi = \Phi$.

v) $\infty \times x = -\infty$ for all $x < 0$.

w) $\infty \times x = \infty$ for all $x > 0$.

x) $-\infty \times 0 = \Phi$.

y) $-\infty \times \Phi = \Phi$.

z) $-\infty \times x = \infty$ for all $x < 0$.

α) $-\infty \times x = -\infty$ for all $x > 0$.

Proof. Let $x \in \mathbb{R}^T$ be arbitrary:

a) Let $x \in \mathbb{R}$ be arbitrary. Take $y \in \mathbb{R}$ such that $y > x$. As $y - x \in \mathbb{R}$ and $y - x > 0$, we have that $\delta < y - x$ for all $\delta \in {}^*\mathbb{R}_0$, that is, $x + \delta < y$ for all $\delta \in {}^*\mathbb{R}_0$. As $y < \omega$ for all $\omega \in {}^*\mathbb{R}^+_\infty$, we have that $x+\delta < y < \omega$ for all $\delta \in {}^*\mathbb{R}_0$ and all $\omega \in {}^*\mathbb{R}^+_\infty$. Thus $x = \{x+\delta;\ \delta \in {}^*\mathbb{R}_0\} < {}^*\mathbb{R}^+_\infty = \infty$. In a similar way, we have that $-\infty < x$.

b) As $x \in \mathbb{R}^T$, we have that there is $A \subset {}^*\mathbb{R}$ such that $x = A$. For all $a \in A$, there is $b \in {}^*\mathbb{R}$ such that $b \leq a$ whence $A < {}^*\mathbb{R}$ does not hold, that is, $x < \Phi$ does not hold. Further, for all $a \in A$, there is $c \in {}^*\mathbb{R}$ such that $a \leq c$ whence ${}^*\mathbb{R} < A$ does not hold, that is, $\Phi < x$ does not hold.

c) $-\Phi = -{}^*\mathbb{R} = {}^*\mathbb{R} = \Phi$.

d) $-(-\infty) = -{}^*\mathbb{R}^-_\infty = {}^*\mathbb{R}^+_\infty = \infty$.

e) $-(\infty) = -{}^*\mathbb{R}^+_\infty = {}^*\mathbb{R}^-_\infty = -\infty$.

f) $0^{-1} = ({}^*\mathbb{R}_0)^{-1} = \{|a|^{-1};\ a \in {}^*\mathbb{R}_0 \setminus \{0\}\} = {}^*\mathbb{R}^+_\infty = \infty$.

g) $\Phi^{-1} = ({}^*\mathbb{R})^{-1} = \{a^{-1};\ a \in {}^*\mathbb{R} \setminus \{0\}\} \cup \{0\} = {}^*\mathbb{R} = \Phi$.

h) $(-\infty)^{-1} = ({}^*\mathbb{R}^-_\infty)^{-1} = \{a^{-1};\ a \in {}^*\mathbb{R}^-_\infty\} \cup \{-a^{-1};\ a \in {}^*\mathbb{R}^-_\infty\} \cup \{0\} = {}^*\mathbb{R}_0 = 0$.

i) The result follows in a similar way to the item h.

j) The result follows of the fact of ${}^*\mathbb{R} + A = {}^*\mathbb{R}$ for all $A \subset {}^*\mathbb{R}$.

k) $\infty+(-\infty) = {}^*\mathbb{R}^+_\infty + {}^*\mathbb{R}^-_\infty = \{a+b;\ a\in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}^-_\infty\} = {}^*\mathbb{R} = \Phi$. In fact, given $x \in {}^*\mathbb{R}$ arbitrary:

- If $x \in {}^*\mathbb{R}^+_\infty$ so $2x \in {}^*\mathbb{R}^+_\infty$, $-x \in {}^*\mathbb{R}^-_\infty$ and $x = 2x+(-x)$ whence $x \in \{a+b;\ a\in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}^-_\infty\}$.
- If $x \in {}^*\mathbb{R}^-_\infty$ so $2x \in {}^*\mathbb{R}^-_\infty$, $-x \in {}^*\mathbb{R}^+_\infty$ and $x = (-x)+2x$ whence $x \in \{a+b;\ a\in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}^-_\infty\}$.
- If $x \notin {}^*\mathbb{R}^-_\infty \cup {}^*\mathbb{R}^+_\infty$ so take $\omega \in {}^*\mathbb{R}^+_\infty$ arbitrary fixed. We have that $x+\omega \in {}^*\mathbb{R}^+_\infty$, $-\omega \in {}^*\mathbb{R}^-_\infty$ and $x = x+\omega+(-\omega)$ whence $x \in \{a+b;\ a\in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}^-_\infty\}$.

l) The result follows in a similar way to the item j.

m) $\infty+\infty = {}^*\mathbb{R}^+_\infty + {}^*\mathbb{R}^+_\infty = {}^*\mathbb{R}^+_\infty = \infty$.

n) $\infty + x = {}^*\mathbb{R}^+_\infty + \{x+\delta;\ \delta \in {}^*\mathbb{R}_0\} = \{y+x+\delta;\ y \in {}^*\mathbb{R}^+_\infty \text{ and } \delta \in {}^*\mathbb{R}_0\} = {}^*\mathbb{R}^+_\infty = \infty$ for all $x \in \mathbb{R}$.

o) The result follows in a similar way to the item k.

p) The result follows in a similar way to the item j.

q) The result follows in a similar way to the item m.

r) The result follows in a similar way to the item n.

s) The result follows of the fact of ${}^*\mathbb{R} \times A = {}^*\mathbb{R}$ for all $A \subset {}^*\mathbb{R}$.

t) $\infty \times 0 = {}^*\mathbb{R}^+_\infty \times {}^*\mathbb{R}_0 = \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}_0\} = {}^*\mathbb{R} = \Phi$. In fact, given $x \in {}^*\mathbb{R}$ arbitrary:

- If $x \in {}^*\mathbb{R}^+_\infty \cup {}^*\mathbb{R}^-_\infty$ so $x^2 \in {}^*\mathbb{R}^+_\infty$, $x^{-1} \in {}^*\mathbb{R}_0$ and $x = x^2 \times x^{-1}$ whence $x \in \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}_0\}$.
- If $x = 0$ so take $\omega \in {}^*\mathbb{R}^+_\infty$ arbitrary. We have that $\omega \in {}^*\mathbb{R}^+_\infty$, $0 \in {}^*\mathbb{R}_0$ and $x = 0 = \omega \times 0$ whence $x \in \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}_0\}$.
- If $x \in {}^*\mathbb{R}_0 \setminus \{0\}$ so $\frac{1}{|x|} \in {}^*\mathbb{R}^+_\infty$, $x|x| \in {}^*\mathbb{R}_0$ and $x = \frac{1}{|x|} \times x|x|$ whence $x \in \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}_0\}$.
- If $x \notin {}^*\mathbb{R}_0 \cup {}^*\mathbb{R}^+_\infty \cup {}^*\mathbb{R}^-_\infty$ so take $\delta \in {}^*\mathbb{R}_0$ such that $\delta > 0$ arbitrary. We have that $\frac{x^2}{|x|\delta} \in {}^*\mathbb{R}^+_\infty$, $\frac{|x|\delta}{x} \in {}^*\mathbb{R}_0$ and $x = \frac{x^2}{|x|\delta} \times \frac{|x|\delta}{x}$ whence $x \in \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}_0\}$.

u) The result follows in a similar way to the item s.

v) Suppose $x < 0$:

- If $x = -\infty$ so $\infty \times x = {}^*\mathbb{R}^+_\infty \times {}^*\mathbb{R}^-_\infty = \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}^-_\infty\} = {}^*\mathbb{R}^-_\infty = -\infty$. In fact, clearly $\{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}^-_\infty\} \subset {}^*\mathbb{R}^-_\infty$. Now if $x \in {}^*\mathbb{R}^-_\infty$ so $\sqrt{-x} \in {}^*\mathbb{R}^+_\infty$, $-\sqrt{-x} \in {}^*\mathbb{R}^-_\infty$ and $x = \sqrt{-x} \times (-\sqrt{-x})$ whence $x \in \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in {}^*\mathbb{R}^-_\infty\}$.
- If $x \in \mathbb{R}$ so $\infty \times x = {}^*\mathbb{R}^+_\infty \times \{x+\delta;\ \delta \in {}^*\mathbb{R}_0\} = \{y(x+\delta);\ y \in {}^*\mathbb{R}^+_\infty \text{ and } \delta \in {}^*\mathbb{R}_0\} = {}^*\mathbb{R}^-_\infty = -\infty$. In fact, let $y \in {}^*\mathbb{R}^+_\infty$ and $\delta \in {}^*\mathbb{R}_0$ arbitrary. Let us see that $y(x+\delta) < z$ for all $z \in \mathbb{R}$. If $z \in \mathbb{R}$ is arbitrary, we take $t \in \mathbb{R}$ such that $t > \frac{z}{x+\delta}$, so $y > t > \frac{z}{x+\delta}$ whence $y(x+\delta) < z$. Thus $y(x+\delta) \in {}^*\mathbb{R}^-_\infty$. Now if $\omega \in {}^*\mathbb{R}^-_\infty$ so $\frac{\omega}{x} \in {}^*\mathbb{R}^+_\infty$, $x \in \{x+\delta;\ \delta \in {}^*\mathbb{R}_0\}$ and $\omega = \frac{\omega}{x} \times x$ whence $\omega \in \{ab;\ a \in {}^*\mathbb{R}^+_\infty \text{ and } b \in \{x+\delta;\ \delta \in {}^*\mathbb{R}_0\}\}$.

w) The result follows in a similar way to the item v.

x) The result follows in a similar way to the item t.

y) The result follows in a similar way to the item s.

z) The result follows in a similar way to the item v.

α) The result follows in a similar way to the item v.

□

Corollary 8. Let $x, y \in \mathbb{R}$ where $x > 0$ and $y < 0$. It follows that:

a) $x \div 0 = \infty$.

b) $y \div 0 = -\infty$.

c) $0 \div 0 = \Phi$.

Proof. Let $x, y \in \mathbb{R}$ where $x > 0$ and $y < 0$:

a) $x \div 0 = x \times 0^{-1} = x \times \infty = \infty$.

b) $y \div 0 = y \times 0^{-1} = y \times \infty = -\infty$.

c) $0 \div 0 = 0 \times 0^{-1} = 0 \times \infty = \Phi$.

□

4 Interpretation of Transreal Arithmetic

Real numbers may be interpreted as vectors oriented in a straight line, whence the operations of real arithmetic are vector transformations: addition is a translation and multiplication is a dilatation. Similarly we can interpret each transreal number as a vector. The transreal number ∞ is a positively oriented vector whose "size" (magnitude) is greater than the size of any real vector. With the construction in the previous section, ∞ is a positively oriented vector whose size is a positive infinite hyperreal number. But ∞ does not have a fixed size. We can see the size of this vector as being similar to the localisation of an electron in its orbital. That is, at each time in which ∞ is operated on, it has some size which is certainly a positive infinite hyperreal number, but not necessarily the same number as in a previous instant (operation). We emphasise that we are not interpreting ∞ as many vectors, that is, an indeterminate vector. Instead ∞ is a unique, specific vector but with variable length. In a similar way we interpret $-\infty$ as a vector oriented negatively whose size is a negative infinite hyperreal number. The number Φ can be understood as a vector whose size assumes a value each time it is operated on, but unlike ∞, the size of Φ does not have the restriction of being greater than any real number. The number Φ is unique, definite and determinate, however its size, as a vector, can be any hyperreal number at each moment in which Φ is operated on. Furthermore, we understand that Φ does not reveal information about its size. That is, at each moment in which Φ is operated on, it has a certain size which is one among all hyperreal numbers, but not necessarily the same number as in a previous instant. Every real number is interpreted as the oriented vector whose size is any hyperreal number in the monad of that real number.

We operate on the transreal numbers by vector transformations but at each time that the operation is performed, the size of the vector is a hyperreal number randomly selected from its "orbital". In an operation, the selected number can be different from one selected in the previous operation, however the selected hyperreal number lies on the specific "orbital" of that transreal number. The "orbital" of nullity is the whole set of hyperreal numbers, the "orbital" of positive infinity is the set of positive infinite hyperreal numbers, the "orbital" of negative infinity is the set of negative infinite hyperreal numbers and the "orbital" of each real number is its monad.

If $x \in \mathbb{R}$, then the addition $\infty + x$ can be interpreted as the translation of x by ∞. Since $x \in \mathbb{R}$ has a finite size and ∞ has some infinite hyperreal size and is positively oriented, $\infty + x$ has an infinite hyperreal size and is positively oriented regardless of the orientation and size of x. Thus $\infty + x = \infty$. The addition $\infty + (-\infty)$ can be interpreted as the translation of a vector, of positive orientation and unknown size but greater than any real number, by a vector of negative orientation of unknown size but greater than any real number (not necessarily equal to the positive vector). Thus $\infty + (-\infty)$ can have any orientation and can get any non negative hyperreal number as size. Therefore $\infty + (-\infty) = \Phi$. The reader can already deduce the interpretation for $\infty + \infty = \infty$ and $\infty + \Phi = \Phi$ and for the other operations of addition and multiplication.

In real numbers, division does not have a vector interpretation, unless the division is obtained by multiplication by the inverse. That is, if $x, y \in \mathbb{R}$ and $y \neq 0$, then $x \div y = x \times y^{-1}$. This can be rewritten as $x \div y = \frac{x}{1} \div \frac{y}{1} = \frac{x}{1} \times \frac{1}{y}$. That is, "division by" is "multiplication by the reciprocal of". The reciprocal of a fraction with numerator zero can be obtained by interpreting the reciprocal as just the reversal of the numerator and denominator, but keeping the sign in the numerator. Thus, in transreal numbers, division by zero is multiplication by $\frac{1}{0}$.

5 Conclusion

We have constructed the transreal numbers as sets of hyperreal numbers in a way that the arithmetical operations between transreal numbers are defined by operations on hyperreal numbers. This construction is one more proof of the consistency of transreal arithmetic.

This construction allows us to give a contextual interpretation for transreal arithmetic as vector transformations. Transreal numbers are vectors with random sizes selected from appropriate sets of hyperreal numbers. The addition operation is a translation and the multiplication operation is a dilatation.

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